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CONFINED SUBGROUPS IN PERIODIC SIMPLE FINITARY LINEAR GROUPS

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Dedicated to the memory of our friend and collaborator Richard E. Phillips

ABSTRACT

A subgroup X of the locally finite group G is said to be **confined**, if there exists a finite subgroup $F \leq G$ such that $X^g \cap F \neq 1$ for all $g \in G$. Since there seems to be a certain correspondence between proper confined subgroups in G and non-trivial ideals in the complex group algebra $\mathbb{C}G$, we determine the confined subgroups of periodic simple finitary linear groups in this paper.

1. Introduction

The motivation for the present paper is to determine the ideal lattices of group algebras of locally finite simple groups. It has been shown in several instances that this lattice can be quite sparse. For example, any group algebra $\mathbb{K}G$ of P. Hall's universal locally finite group $G = \mathrm{ULF}$ just contains the trivial ideals $\mathbb{K}G$, 0, and $\omega(\mathbb{K}G)$ (augmentation ideal) [3]. And in fact it is an old question due to

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I. Kaplansky [15] for which groups G and which fields $\mathbb{K}$ the augmentation ideal $\omega(\mathbb{K}G)$ is simple. Every such group G must necessarily be simple.

Kaplansky's question was the starting point for a research programme begun by A. E. Zalesskiĭ to determine the ideal lattices of complex group algebras of locally finite simple groups G. This is also linked with character theory of finite groups, since ideals in $\mathbb{C}G$ correspond naturally with certain systems of irreducible complex characters of the finite subgroups of G [30].

At present, every locally finite simple group can be sorted into one of the following classes: finite groups, infinite linear groups, non-linear finitary linear groups, groups of 1-type, groups of p-type (where p is a prime), and groups of ∞-type. Here a group is said to be **finitary linear** if it acts faithfully on an infinite-dimensional vector space V in such a way that the fixed-point space of each element has finite codimension in V. Locally finite simple groups of **1-type**, **p-type**, or **∞-type** are defined via Kegel covers as certain non-finitary direct limits of finite groups (see Section 2).

Combining results due to S. Delcroix, B. Hartley, and A. E. Zalesskiĭ, we will show in Section 2 that over fields of characteristic zero, group algebras of locally finite simple groups of p-type or of ∞-type admit only the trivial ideals. The same is known for infinite periodic simple linear groups ([11], Theorem B and [13], Theorem 1.1), and hence also for periodic simple finitary linear groups with natural action on a vector space over an infinite field (see Theorem 4.5). Therefore it becomes interesting to study ideal lattices in group algebras of non-linear finitary linear periodic simple groups with natural action on a vector space over a **finite** field. These groups have been classified recently by J. I. Hall [10] and are either alternating groups or natural generalizations of the finite-dimensional classical groups (see Section 3). For alternating groups and for special transvection groups the ideal lattices in question have been determined [5], [12]. In the present article we are therefore mainly concerned with the remaining cases of finitary symplectic, finitary unitary, and finitary orthogonal groups.

Let $\mathbb{K}$ be a field of characteristic zero. By [13] non-trivial ideals in $\mathbb{K}G$ give rise to proper confined subgroups in the locally finite simple group G. Here a subgroup X of G is said to be **confined**, if there exists a finite subgroup $F \leq G$ such that $X^g \cap F \neq 1$ for all $g \in G$. It is conjectured that there is a certain correspondence between proper confined subgroups in G and non-trivial ideals in $\mathbb{C}G$. Hence we determine the confined subgroups in periodic finitary linear simple groups in the first place. This shall provide a basis for the determination of ideal lattices in a subsequent paper. Our main results are as follows.

THEOREM A: *Let G be a classical finitary linear group of isometries relative to a non-degenerate symplectic, unitary, or quadratic form. Suppose that G admits its natural finitary representation on the infinite-dimensional vector-space V over the finite field $\mathbb{F}$, and that X is a confined subgroup in G. Then there exists a unique minimal X-invariant subspace W of finite codimension in V, and the following hold.*

(a) *If $\operatorname{char}\mathbb{F} = 2$, and if G is a finitary symplectic group, then the symplectic bilinear form can be related to a quadratic form q on V with the property that $X \cap H$ has finite index in $N_H(W)$, where H denotes the finitary orthogonal subgroup of G preserving q.*

(b) *In all other cases, X has finite index in $N_G(W)$.*

Over fields of characteristic 2, every finitary orthogonal group with non-degenerate but defective quadratic form is isomorphic to the finitary symplectic group which acts naturally on the quotient of the original space modulo the radical of the associated bilinear form. Therefore all possibilities of classical finitary linear groups of isometries are covered by Theorem A. And in particular all finitary orthogonal groups considered in this paper will have a non-defective quadratic form.

Note also that the various subgroups of finite index in normalizers as described in Theorem A, as well as the subgroups $\mathrm{T}_{\mathbb{F}}(\Gamma, U)$ in Theorem B below, are indeed confined in G (see Section 4). Confined subgroups X of finite index in $N_G(W)$ do of course contain almost the whole centralizer $C_G(V/W)$. If $V = W \oplus W^\perp$, then $C_G(V/W) = C_G(W^\perp) \leq N_G(W^\perp) = N_G(W)$. In this situation we can in fact show that $C_G(W^\perp) \leq X \leq N_G(W^\perp)$ resp. $C_H(W^\perp) \leq X \cap H \leq N_H(W^\perp)$ (Corollary 7.7).

In principle, we also allow $W = V$. In part (a) of Theorem A this implies that the finitary orthogonal subgroups of finitary symplectic groups over finite fields of characteristic 2 are confined.

THEOREM B: *Let X be a confined subgroup of the special transvection group $G = \mathrm{T}_{\mathbb{F}}(\Delta, V)$ over the finite field $\mathbb{F}$. Then there exist subspaces $U \leq V$ and $\Gamma \leq \Delta$ of finite codimensions, satisfying $\operatorname{ann}_\Gamma U = 0$ and $\operatorname{ann}_U \Gamma = 0$, such that $\mathrm{T}_{\mathbb{F}}(\Gamma, U) \leq X$.*

After submission of the present paper, A. E. Zalesskiĭ made the authors aware of an unpublished handwritten manuscript by B. Hartley, in which Theorem B is proved too.

Since every infinite-dimensional vector space V of cardinality $\aleph$ over a finite field has $2^{\aleph}$ subspaces of finite codimension, it follows that in every classical finitary linear group G with natural representation on V the normalizers of subspaces of finite codimension in V form a family of $2^{\aleph}$ confined subgroups (Corollary 5.3).

In view of Theorem A, and in view of the fact that — due to [12] — there is a natural one-to-one correspondence between non-trivial ideals in complex groups algebras of special transvection groups G on the one hand and finite-dimensional subspaces of the natural module of G on the other hand, we formulate the following conjecture.

CONJECTURE: *The non-trivial ideals in complex group algebras of classical finitary linear groups G of isometries are in one-to-one correspondence with the isomorphism types of finite-dimensional subspaces of the natural module for G.*

The authors intend to study this relationship in a subsequent paper.

The structure of the present paper is as follows. In Section 2 we treat locally finite simple groups of p-type or ∞-type. In Section 3 we recall the definition and some basic properties of the classical finitary linear groups. In Section 4 we show that proper confined subgroups X in classical finitary linear groups G exist only over finite fields, and that the subgroups described in Theorems A and B are indeed confined. Section 5 is devoted to the study of the action of confined subgroups X on the natural module of the classical finitary linear group G. In Section 6 some cohomological results are derived which will be needed to split certain groups and modules in the proofs of Theorems A and B. The proofs of the main results will then be given in Section 7. Finally, Section 8 contains the discussion of a couple of questions about confined subgroups raised by A. E. Zalesskiĭ.

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2. Groups of p-type or ∞-type

Let G be a locally finite group. A **Kegel cover** in G is a family $\{(G_i, M_i) \mid i \in I\}$, where each G_i is a finite subgroup of G, and where M_i is a maximal normal subgroup in G_i, such that for every finite subgroup F of G there exists some $i \in I$ satisfying $F \leq G_i$ and $F \cap M_i = 1$. The sections G_i/M_i are called the **Kegel factors** of the Kegel cover. It was proved in [16] that every locally finite simple group has a Kegel cover.

From [4], Theorem 3.29, every locally finite simple group G is either finitary linear, of 1-type, of p-type (where p is a prime), or of ∞-type. Here G is said to be of **1-type** resp. of **p-type**, if every Kegel cover of G has an alternating Kegel factor resp. a Kegel factor which is a finite classical group defined over a field of characteristic p.

We shall show now that certain group algebras of locally finite simple groups of p-type or ∞-type have no non-trivial ideals.

THEOREM 2.1: *Let G be an infinite locally finite simple group with a Kegel cover whose factors are projective special linear groups. Suppose that $\mathbb{K}$ is a field of characteristic zero. If the group ring $\mathbb{K}G$ has a non-trivial ideal, then G is finitary linear.*

Proof: Let $\{(G_i, M_i) \mid i \in I\}$ be the Kegel cover. Assume that the group algebra $\mathbb{K}G$ has a non-trivial ideal. From [13], Theorem 1.1, the group G contains a proper confined subgroup X then, that is, there exists a finite subgroup F in G such that $X^g \cap F \neq 1$ for all $g \in G$. Without loss we may assume that $F \leq G_i$ and $F \cap M_i = 1$ for all i. Let $G_i/M_i \cong \mathrm{PSL}_{\mathbb{F}_i}(V_i)$. Consider the action of G_i on the set Ω_i of right cosets of $X \cap G_i$ in G_i via right translation. Since $X \cap G_i$ is confined in G_i with respect to F, the group F has no regular orbit on Ω_i. Therefore the projective degrees of some non-trivial element $f \in F$ in its action on the modules V_i are uniformly bounded by a function of $|F|$ (see [4], Proposition 3.25). The diagonal homomorphism of G into the ultraproduct of the quotients G_i/M_i with respect to a suitable ultrafilter now yields a projective representation of G on the corresponding ultraproduct V of the V_i such that f has finite projective degree. Since G is simple, and since the normal subgroup $\mathrm{FGL}(V)$ in $\mathrm{GL}(V)$ has trivial intersection with the group of scalars, we thus obtain a faithful finitary representation of G. (This nice kind of ultraproduct argument was by the way developed in J. I. Hall [8].) ∎

Since by [4], Theorem 4.27 resp. Lemma 3.27 every group of p-type resp. of ∞-type has a Kegel cover as in Theorem 2.1, the following corollary is an immediate consequence.

COROLLARY 2.2: *If G is a non-finitary locally finite simple group of p-type or of ∞-type, and if $\mathbb{K}$ is a field of characteristic zero, then $\mathbb{K}G$ has only the trivial ideals.*

Because of the above corollary the remainder of the present article is devoted to the study of periodic simple finitary linear groups.

3. Classical finitary linear groups

Due to J. Hall [10], every non-linear finitary linear periodic simple group is one of the following.

(1) an infinite alternating group $\mathrm{Alt}(\Omega)$,
(2) a special transvection group $\mathrm{T}_{\mathbb{F}}(\Delta, V)$,
(3) a finitary symplectic group $\mathrm{FSp}_{\mathbb{F}}(V, b)$,
(4) a finitary special unitary group $\mathrm{FSU}_{\mathbb{F}}(V, b)$,
(5) a finitary orthogonal group $\mathrm{F}\Omega_{\mathbb{F}}(V, q)$.

Here $\mathbb{F}$ is always a locally finite field, V is an infinite-dimensional $\mathbb{F}$-vector space with non-degenerate form b resp. q, and Δ is a subspace of the dual space V^* satisfying $\mathrm{ann}_V\,\Delta = 0$.

The ideal lattice of $\mathbb{F}(\mathrm{Alt}(\Omega))$ for fields $\mathbb{F}$ of characteristic zero can be derived from [5] (see also [25], [28]). Moreover the confined subgroups of $\mathrm{Alt}(\Omega)$ are described in [26]. We shall therefore restrict our attention to groups of types (2) – (5) above, which we call **classical finitary linear** groups. In this section we collect those features of these groups which will be essential in this paper. For further information the reader is referred to [10], Section 2.

Let V be an infinite-dimensional vector space over the locally finite field $\mathbb{F}$. For every $x \in V$ and every $\varphi \in V^*$ with $x\varphi = 0$, the **transvection** $\tau_{\varphi,x}\colon V \to V$ is defined via $v\tau_{\varphi,x} = v + (v\varphi)x$ for all $v \in V$. For $\Delta \leq V^*$ the **special transvection group** $\mathrm{T}_{\mathbb{F}}(\Delta, V)$ is given by

$$\mathrm{T}_{\mathbb{F}}(\Delta, V) = \langle \tau_{\varphi,x} \mid \varphi \in \Delta,\ x \in V,\ x\varphi = 0 \rangle.$$

Special transvection groups are generalizations of special linear groups: $\mathrm{T}_{\mathbb{F}}(V^*, V)$ consists of all finitary transformations of V with determinant 1. The group $\mathrm{T}_{\mathbb{F}}(\Delta, V)$ is simple whenever $\mathrm{ann}_V\,\Delta = 0$. Clearly, we call V the **natural module** for $\mathrm{T}_{\mathbb{F}}(\Delta, V)$. Moreover Δ will be referred to as the **conatural module** for $\mathrm{T}_{\mathbb{F}}(\Delta, V)$.

Let the vector space V be equipped with a **symplectic** bilinear form $b\colon V \times V \to \mathbb{F}$, that is, $b(v, v) = 0$ for all $v \in V$. Such a form is **non-degenerate** if its radical $V^{\perp}$ is trivial. The **finitary symplectic group** $\mathrm{FSp}_{\mathbb{F}}(V, b)$ is the group of all finitary isometries of b. It is simple whenever b is non-degenerate, and

$$\mathrm{FSp}_{\mathbb{F}}(V, b) = \langle \tau_{\varphi,x} \mid x \in V, \text{and } \varphi = b(\cdot, ax) \text{ for some } a \in \mathbb{F} \rangle.$$

Suppose next, that the field $\mathbb{F}$ admits an involutory automorphism α, and that a **unitary** sesquilinear form $b\colon V \times V \to \mathbb{F}$ is given by $b(u, v) = b(v, u)^{\alpha}$ for all $u, v \in V$. Again such a form is **non-degenerate** if its radical $V^{\perp}$ is trivial.

The **finitary unitary group** $\mathrm{FSU}_{\mathbb{F}}(V,b)$ is the group of all finitary isometries of b with determinant 1. It is simple whenever b is non-degenerate, and then $\mathrm{FSU}_{\mathbb{F}}(V,b)$ is generated by all transvections $\tau_{\varphi,x}$ satisfying $\varphi = b(\cdot, ax)$ for some isotropic vector $x \in V$ and for some $a \in \mathbb{F}$ with $a^\alpha = -a$.

Finally, V may admit a **quadratic** form, that is, a map $q\colon V \to \mathbb{F}$ satisfying $q(av) = a^2 q(v)$ for all $a \in \mathbb{F}$, $v \in V$. The associated **orthogonal** bilinear form $b\colon V \times V \to \mathbb{F}$ is then defined via $b(u,v) = q(u+v) - q(u) - q(v)$ for all $u, v \in V$. We define $V^\perp$ with respect to b. Now q is said to be **non-degenerate** if the radical $\{v \in V^\perp \mid q(v) = 0\}$ is trivial. If $\operatorname{char}\mathbb{F} \neq 2$, then b determines q, and so the radical coincides with $V^\perp$. If $\operatorname{char}\mathbb{F} = 2$, then a non-degenerate quadratic form q can have a degenerate associated orthogonal form b, in which case we call q **defective**. The **finitary orthogonal group** $\mathrm{F}\Omega_{\mathbb{F}}(V,q)$ is the derived subgroup of the group of all finitary isometries of q. It is simple whenever q is non-degenerate. In the defective characteristic 2 case, $\mathrm{F}\Omega_{\mathbb{F}}(V,q) \cong \mathrm{FSp}_{\mathbb{F}}(V/V^\perp, \bar{b})$ where $\bar{b}$ is the bilinear form induced from b on $V/V^\perp$. We may therefore always assume that our orthogonal forms are non-degenerate. In general, the group $\mathrm{F}\Omega_{\mathbb{F}}(V,q)$ is generated by the so-called **Siegel elements** σ, which are defined by the following conditions:

(i) σ is an isometry of q,

(ii) $(\sigma - 1)^2 = 0$, and

(iii) $V(\sigma - 1)$ is totally singular of dimension 2.

The Siegel elements, resp. the transvections generating the classical finitary linear groups in cases (2) – (4) above, are called **root elements**.

Every countably-dimensional vector space V over a locally finite field $\mathbb{F}$, which admits a non-degenerate symplectic, unitary or orthogonal form b, has a basis $\mathcal{B} = \{v_n \mid n \in \mathbb{N}\}$ such that V is the orthogonal sum of the hyperbolic planes $W_n = \langle v_{2n}, v_{2n+1}\rangle$ (see Corollary 4.3). We call such a basis of pairwise orthogonal hyperbolic pairs a **standard basis**. The full finitary linear group of isometries of (V,b) is **stable** with respect to $\mathcal{B}$, that is, every finite subgroup of G fixes all but finitely many vectors in $\mathcal{B}$. Note also that every basis $\mathcal{B}$ in the countably-dimensional vector space V leads to a stable general finitary linear group $\mathrm{GL}_{\mathbb{F}}(V,\mathcal{B})$, and to a stable special finitary linear group $\mathrm{SL}_{\mathbb{F}}(V,\mathcal{B})$, which is a finitary transvection group (see [10], p. 152). In fact, every irreducible subgroup of $\mathrm{FGL}_{\mathbb{F}}(V)$, which is a **countable** finitary transvection group, is the stable special finitary linear group with respect to a suitable basis $\mathcal{B}$ of V (see [10], p. 154). We will also then call $\mathcal{B}$ a standard basis.

4. Existence of proper confined subgroups

In this section we will prove that classical finitary linear groups contain proper confined subgroups if and only if they are defined over a finite field. We begin by mentioning two preparatory lemmata.

LEMMA 4.1: *Let V be a vector space over the field $\mathbb{F}$, equipped with a non-degenerate symplectic, unitary or orthogonal form b. Suppose that U is a non-degenerate finite-dimensional subspace of V. Then $V = U \oplus U^{\perp}$ and $U^{\perp\perp} = U$.*

Proof: Let $d = \dim_{\mathbb{F}} U$. Since $U^{\perp}$ is an intersection of d hyperplanes in V, we have $\dim_F V/U^{\perp} \leq d$. But $U \cap U^{\perp} = 0$ by non-degeneracy of U. ∎

LEMMA 4.2: *Let V be an infinite-dimensional vector space over the field $\mathbb{F}$, equipped with a non-degenerate symplectic, unitary or orthogonal form b. Suppose in addition that the field $\mathbb{F}$ is perfect in the case when b is orthogonal and $\operatorname{char} \mathbb{F} = 2$. Then every finite-dimensional subspace U of V is contained in a non-degenerate finite-dimensional subspace W of dimension $2 \cdot \dim_{\mathbb{F}} U$, which is an orthogonal sum of hyperbolic planes.*

Hyperbolic planes are generated by hyperbolic pairs. Note that in the orthogonal case a hyperbolic pair v, w is defined via $b(v, w) = 1$ and $q(v) = 0 = q(w)$, where q denotes the corresponding quadratic form.

Proof of Lemma 4.2: Let $\{v_1, \ldots, v_k\}$ be a basis of U. By induction over k, the subspace $\langle v_2, \ldots, v_k \rangle$ is contained in a subspace W of V, which is an orthogonal sum of $k-1$ hyperbolic planes. Now $V = W \oplus W^{\perp}$, and we can consider the component w_1 of v_1 along $W^{\perp}$. It suffices to embed w_1 into a hyperbolic plane in $W^{\perp}$.

Suppose first that the characteristic of the underlying field is odd when b is orthogonal. If w_1 is isotropic, then we choose some $v \in W^{\perp} \setminus \langle w_1 \rangle^{\perp}$. The subspace $\langle w_1, v \rangle$ is non-degenerate, and hence a hyperbolic plane by [27], Lemma 7.3. Next, let w_1 be non-isotropic. The infinite-dimensional non-degenerate space $W^{\perp} \cap \langle w_1 \rangle^{\perp}$ contains a hyperbolic plane H, and there exists an isometry $\langle w_1 \rangle \to H$. In this case it follows from Witt's Theorem (see [27], Theorem 7.4) that some isometry in the classical finitary linear group on $W^{\perp}$ carries w_1 into H.

It remains to consider the case when b is the associated orthogonal form of a quadratic form q in characteristic 2. Fix $v \in W^{\perp} \setminus \langle w_1 \rangle^{\perp}$. Since the field $\mathbb{F}$ is perfect, the non-degenerate space $\langle W, w_1, v \rangle^{\perp}$ contains a vector v' with $q(v') =$

$q(v)$. It follows that $v + v'$ is singular and $b(w_1, v + v') \neq 0$. Then again [27], Lemma 7.3 shows that w_1 and $v + v'$ generate a hyperbolic plane. ∎

COROLLARY 4.3: *Every countably-dimensional subspace of a vector space V with non-degenerate symplectic, unitary or orthogonal form is contained in a countably-dimensional orthogonal sum of hyperbolic planes in V. (Here we suppose in addition that the underlying field is perfect in the case of an orthogonal form in characteristic 2.)*

Proof: Let $\{v_i \mid i \in \mathbb{N}\}$ be a basis of the subspace U. Apply Lemma 4.2 recursively to construct a hyperbolic plane H_{i+1} in $V_i^\perp = (\sum_{j\leq i} H_j)^\perp$ such that the projection of v_i onto $V_i^\perp$ with respect to the decomposition $V = V_i \oplus V_i^\perp$ is contained in H_{i+1}. Then U is contained in the orthogonal sum of the H_i. ∎

LEMMA 4.4: *Let V be a vector space over the field $\mathbb{F}$, equipped with a non-degenerate symplectic, unitary or orthogonal form b. Suppose in addition that the field $\mathbb{F}$ is perfect in the case when b is orthogonal and $\operatorname{char}\mathbb{F} = 2$. If U is a finite-dimensional subspace of V, then $\dim_{\mathbb{F}} V/U^\perp = \dim_{\mathbb{F}} U$ and $U^{\perp\perp} = U$. Moreover, $(U \cap W)^\perp = U^\perp + W^\perp$ for every finite-dimensional subspace W of V.*

Proof: Let B be an $\mathbb{F}$-basis of U. Then $U^\perp$ is the intersection of the hyperplanes $\{v \in V \mid b(v,u) = 0\}$ $(u \in B)$. Therefore $\dim_{\mathbb{F}} V/U^\perp \leq \dim_{\mathbb{F}} U$. On the other hand, U is contained in a non-degenerate finite-dimensional subspace L of V by Lemma 4.2, and so $L/L \cap U^\perp \cong L + U^\perp/U^\perp \leq V/U^\perp$, whence $\dim_{\mathbb{F}} U = \dim_{\mathbb{F}} L/L \cap U^\perp \leq \dim_{\mathbb{F}} V/U^\perp$.

Clearly $U \leq U^{\perp\perp}$. Consider a direct complement C to $U^\perp$ in V. Then $U^{\perp\perp} \cong U^{\perp\perp}/(U^\perp + C)^\perp = U^{\perp\perp}/U^{\perp\perp} \cap C^\perp \cong U^{\perp\perp} + C^\perp/C^\perp \leq V/C^\perp$. It follows that $\dim_{\mathbb{F}} U^{\perp\perp} \leq \dim_{\mathbb{F}} V/C^\perp = \dim_{\mathbb{F}} C = \dim_{\mathbb{F}} V/U^\perp = \dim_{\mathbb{F}} U$.

Clearly, $U^\perp + W^\perp \leq (U \cap W)^\perp$. Conversely, for every non-degenerate finite-dimensional subspace L of V containing $U + W$, we have $L \cap (U \cap W)^\perp = (L \cap U^\perp) + (L \cap W^\perp) \leq U^\perp + W^\perp$. And so the last assertion follows from Lemma 4.2. ∎

For the proofs in this section we need to establish certain local systems in classical finitary linear groups G. In the case when G is a group of isometries, we let $\mathcal{G}$ be the set of classical linear subgroups $G_U = N_G(U) \cap C_G(U^\perp) \leq G$, where U ranges over all finite-dimensional non-degenerate subspaces U of V. Lemmata 4.1 and 4.2 ensure that $\mathcal{G}$ is a local system in G.

In the case when $G = \mathrm{T}_{\mathbb{F}}(\Delta, V)$ is a special transvection group, we also have the conatural action of G on V^*. For every finite subgroup F of G there exist finite-dimensional subspaces $U \leq V$ and $\Gamma \leq \Delta$ satisfying $[V, F] = [U, F] \leq U$ and $[\Delta, F] = [\Gamma, F] \leq \Gamma$, as well as $\mathrm{ann}_\Gamma U = 0$ and $\mathrm{ann}_U \Gamma = 0$ (see [10], p. 153). We write $\Gamma^\perp = \mathrm{ann}_V \Gamma$. Then F is contained in the subgroup

$$G_{\Gamma,U} = N_G(U) \cap C_G(\Gamma^\perp) \cong \mathrm{T}_{\mathbb{F}}(\Gamma, U) \cong \mathrm{T}_{\mathbb{F}}(U^*, U) \cong \mathrm{SL}_{\mathbb{F}}(U)$$

of G. And so we let $\mathcal{G}$ denote the local system of all these groups $G_{\Gamma,U}$.

Theorem 4.5: *Let G be a classical finitary linear group defined over an infinite locally finite field $\mathbb{F}$. Then G has no proper confined subgroups, and hence the ideal lattice in $\mathbb{K}G$ is trivial for any field $\mathbb{K}$ of characteristic zero.*

This result was already observed by A. E. Zalesskiĭ in [29], Proposition 10, modulo [29], Conjecture 5 which was proved later on by D. Gluck (see [11], p. 303).

Proof of Theorem 4.5: Let V denote the natural $\mathbb{F}G$-module. Suppose that X is a confined subgroup of G with respect to the finite subgroup F, that is, $X^g \cap F \neq 1$ for all $g \in G$. Consider the local system $\mathcal{G}$ above. Choose $H = G_U$ (resp. $H = G_{\Gamma,U}$) such that $F \leq H$ and $[V, F] < U$. Then F intersects the center Z of H trivially. Hence either $(X \cap H)Z/Z$ is confined in H/Z with respect to FZ/Z, or $X \cap H \leq Z$. Now [11], Theorem B implies that $X \cap H$ is either central in H or coincides with H.

From passing to a local subsystem of $\mathcal{G}$ we may assume that just one of these two alternatives occurs throughout. In the first case X is central in G, whence $X = 1$, a contradiction. In the second case we obtain $X = G$. ∎

The above theorem allows us to restrict our attention to classical finitary linear groups defined over **finite** fields. These groups contain proper confined subgroups.

Proposition 4.6: *Let G be a classical finitary linear group of isometries, defined over a finite field $\mathbb{F}$, and let V denote the natural $\mathbb{F}G$-module. Suppose further that W is a subspace of finite codimension in V. Then every subgroup of finite index in $N_G(W)$ is confined in G.*

Note that overgroups of confined subgroups are confined too.

Proof of Proposition 4.6: Let X be a subgroup of finite index n in $N_G(W)$. From Lemma 4.2, there exists a non-degenerate subspace U in V, which is an orthogonal sum of $d+\ell$ hyperbolic planes, where d denotes the codimension of W in V, and where ℓ is so large that every finite classical group with natural module of dimension 2ℓ contains at least $n+1$ elements. We choose $F = G_U = N_G(U) \cap C_G(U^{\perp})$. For any $g \in G$, the dimension of $Wg \cap U$ is at least $d + 2\ell$. Therefore $Wg \cap U$ contains a non-degenerate subspace Y of dimension 2ℓ. It follows that $G_Y \leq F \cap N_G(Wg) = F \cap N_G(W)^g$ contains at least $n+1$ elements. Two of them lie in a common coset of X^g in $N_G(U)^g$, whence $F \cap X^g$ is non-trivial, as required. ∎

For finitary symplectic groups over fields of characteristic 2 we can sharpen Proposition 4.6 as follows.

PROPOSITION 4.7: *Let G be a finitary symplectic group, defined over a finite field $\mathbb{F}$ of characteristic 2, and let V denote the natural $\mathbb{F}G$-module, with symplectic from b. Suppose further that W is a subspace of finite codimension in V, and that H is the finitary orthogonal subgroup of G, which is defined with respect to some quadratic form q on V, with associated bilinear form b. Then every subgroup of finite index in $N_H(W)$ is confined in G.*

Proof: We can proceed literally as in the proof of Proposition 4.6, noting that for every $g \in G$, the conjugate H^g is the finitary orthogonal group with respect to the quadratic form $\widehat{q}$, defined via $\widehat{q}(v) = q(vg^{-1})$ for all $v \in V$ (with associated bilinear form b). ∎

PROPOSITION 4.8: *Let $G = \mathrm{T}_{\mathbb{F}}(\Delta, V)$ be a special transvection group over a finite field $\mathbb{F}$. Suppose that $U \leq V$ and $\Gamma \leq \Delta$ are subspaces of finite codimensions. Then $\mathrm{T}_{\mathbb{F}}(\Gamma, U)$ is confined in G.*

Proof: Choose finite-dimensional subspaces $\Delta_0 \leq \Delta$ and $W_0 \leq V$ such that $\mathrm{ann}_{\Delta_0} W_0 = 0$ and $\mathrm{ann}_{W_0} \Delta_0 = 0$, and such that $\dim_{\mathbb{F}} \Delta_0 \geq 2 + \dim_{\mathbb{F}} \Delta/\Gamma$ and $\dim_{\mathbb{F}} W_0 \geq 1 + \dim_{\mathbb{F}} V/U$. Let $F = G_{\Delta_0, W_0} = N_G(W_0) \cap C_G(\Delta_0^{\perp})$, and consider any $g \in G$. There exists a non-zero vector $w \in W_0 g \cap U$, and $\mathrm{ann}_{\Delta} w$ has codimension at most 1 in Δ. Hence there exists a non-zero $\varphi \in \Delta_0 g \cap \mathrm{ann}_{\Gamma} w$. Therefore $F^g \cap \mathrm{T}_{\mathbb{F}}(\Gamma, U)$ contains the transvection $\tau_{\varphi,w}$. ∎

We can also relate confined subgroups of primitive finitary linear groups G to confined subgroups of their derived subgroup G'.

PROPOSITION 4.9: *Let V be a vector space over the finite field $\mathbb{F}$, and suppose that G is a finitary linear group of isometries of a non-degenerate form on V which* *contains the corresponding classical finitary linear group. Then the subgroup X of G is confined in G if and only if $X \cap G'$ is confined in G'.*

Proof: Suppose first that $X \cap G'$ is confined in G', with respect to the finite subgroup $F \leq G'$. For every $g \in G$ there exists $x \in G$ such that $[F, g^x] = 1$. It follows that $F^g = F^{[x,g]}$. Note that $[x, g] \in G'$. And so this shows that $X \cap G'$ is confined in G with respect to F. But then also the overgroup X of $X \cap G'$ is confined in G with respect to F.

Conversely, let X be confined in G with respect to the finite subgroup $F \leq G$. Since the field $\mathbb{F}$ is finite, the kernel K of the determinant map $G \to \mathbb{F}$ has finite index n in G. From consideration of the action of F on V it is easy to find G-conjugates $F_1, \ldots, F_{n+1}$ of F with pairwise trivial intersection. Consider $\widehat{F} = \langle F_1, \ldots, F_{n+1} \rangle$. For every $g \in G$, the intersection $X \cap \widehat{F}^g$ contains $n+1$ non-trivial elements, because $X \cap F_i^g$ is non-trivial. Hence $X \cap \widehat{F}^g$ also contains a non-trivial element of determinant 1. This shows that $X \cap K$ is confined in K with respect to the finite subgroup $F \cap K$. In most cases, K coincides with G'. In all other cases the above argument can be applied with the canonical epimorphism $G \to G/G'$ in place of the determinant map. ∎

5. The action of confined subgroups

It is our next aim to find out how confined subgroups X of classical finitary linear groups G act on the natural G-module. We first study the possible shape of an X-composition series in G.

PROPOSITION 5.1: *Let G be a classical finitary linear group, defined over the field $\mathbb{F}$, and let V denote the natural $\mathbb{F}G$-module. Suppose that X is a confined subgroup of G. Then there exists a constant c (depending on X) such that every X-invariant subspace of V has finite dimension or codimension, bounded by c.*

Proof: Let X be confined with respect to the finite subgroup F of G, and let W be an X-invariant subspace of V. Suppose first that G is a group of isometries of a non-degenerate form b on V. By Lemma 4.2 there exists a non-degenerate subspace U in V, which is a direct sum of n pairwise orthogonal hyperbolic planes, such that $[V, F] \leq U$ and $[U^\perp, F] = 0$. Choose $c = 4n + 2$. Let $W_0 = W \cap W^\perp$ denote the radical of W, and choose non-degenerate subspaces W_i such that $W = W_0 \oplus W_1$ and $W^\perp = W_0 \oplus W_2$.

Assume first that $\dim_{\mathbb{F}} W_0 = m \geq 2n + 2$. Arguing as in the proof of Lemma 4.2 we can recursively choose vectors $s_1, t_1, \ldots, s_m, t_m \in V$ in this order, such that $s_i \in W_0$, and such that $H_i = \langle s_i, t_i \rangle$ is a hyperbolic plane orthogonal to $H_1 \oplus \ldots \oplus H_{i-1}$ for all i (where in addition, in case b is the associated orthogonal form of a quadratic form q on V, we require that $q(t_i) = 0$). For each $i \leq n$, the space $Z_i = \langle t_{2i+1} + s_{2(i+1)}, t_{2(i+1)} \rangle$ is a hyperbolic plane with the property that every non-zero vector in Z_i has non-trivial projection onto $T = \langle t_1, \ldots, t_m \rangle$ with respect to the decomposition $H = W_0 \oplus T$. The same holds for the orthogonal sum $Z = Z_1 \oplus \ldots \oplus Z_n$. It follows that $Z \cap W_0^\perp = Z \cap (T \oplus W_0) \cap W_0^\perp \leq Z \cap W_0 = 0$.

Using Witt's Theorem (see [27], Theorem 7.4), choose $g \in G$ such that $Ug = Z$, and consider $x \in X \cap F^g$. From $Ug \cap W_0 = Z \cap W_0 = 0$ we obtain $V = (Ug)^\perp + {W_0}^\perp$, and this yields $[V, x] = [(Ug)^\perp + {W_0}^\perp, x] = [{W_0}^\perp, x] \leq {W_0}^\perp \cap Ug \leq {W_0}^\perp \cap Z = 0$. But then $X \cap F^g$ is trivial, in contradiction to confinedness of X with respect to F. This contradiction shows that $\dim_{\mathbb{F}} W_0 \leq 2n + 1$.

Assume next that $\dim_{\mathbb{F}} W_i \geq 2n + 2$ for $i = 1, 2$. Then both W_1 and W_2 contain an isometric copy of U. Hence we can find $g \in G$ such that Ug embeds into $W_1 \oplus W_2$ in such a way, that every non-zero vector in Ug projects non-trivially onto both W_1 and W_2. In particular, $Ug \cap W = 0 = Ug \cap W^\perp$. Since U is finite-dimensional, there exists a finite-dimensional subspace R in W such that $Ug \cap R^\perp = 0$. Note that $Ug \cap R \subseteq Ug \cap W = 0$. Moreover, $[R, X \cap F^g] \subseteq W \cap Ug = 0$, whence R and $R^\perp$ are $(X \cap F^g)$-invariant. And so we reach the same contradiction as before with R in place of W_0. This contradiction shows that $\dim_{\mathbb{F}} W_i \leq 2n + 1$ for $i = 1$ or $i = 2$.

It follows from the above that every finite-dimensional X-invariant subspace of V has dimension at most c. Assume now that W is an infinite-dimensional X-invariant subspace of V with $\dim_{\mathbb{F}} V/W > c$. Then $\dim_{\mathbb{F}} W^\perp \leq c$. Let D be a direct complement to $W + W^\perp$ in V. Since both W_1 and $W_2 \oplus D$ contain an isometric copy of U, there exists $g \in G$ such that $Ug \cap W = 0 = Ug \cap W^\perp$. In particular, $[W, X \cap F^g] \subseteq W \cap Ug = 0$. The vector space V carries a vector space topology where the subspaces $L^\perp$ with $\dim_{\mathbb{F}} L < \infty$ form a basis of open (and closed) neighbourhoods of the point 0 (see [6], Section I.8 for further details). This topology is Hausdorff, since V is non-degenerate. And every element from G is a homeomorphism of V. Therefore $X \cap F^g$ does not only act trivially on W, but also on the closure $W^{\perp\perp}$ of W in V. But then $X \cap F^g$ acts trivially on $W^{\perp\perp} + Ug^\perp = V$, in contradiction to confinedness of X.

Suppose finally that $G = \mathrm{T}_{\mathbb{F}}(\Delta, V)$ is a special transvection group. Without loss we may assume that $F = G_{\Gamma, U}$ for certain finite-dimensional subspaces $U \leq V$ and $\Gamma \leq \Delta$. Note that $V = \Gamma^\perp \oplus U$. Choose $c = \dim_{\mathbb{F}} U$.

Assume that $\dim_F W > c$ and $\dim_F V/W > c$. There exists $g \in G$ such that

$U \leq Wg$. Now $Wg = (Wg \cap \Gamma^\perp) \oplus U$, and $Wg \cap \Gamma^\perp$ has codimension larger than c in $\Gamma^\perp$. Choose $h \in G$ such that h acts like the identity on $\Gamma^\perp$, and such that $[u,h] \in \Gamma^\perp \setminus Wg$ for every non-zero $u \in U$. Then $V = \Gamma^\perp + Wgh$ and $Wgh \cap U = 0$. For $x \in F \cap X^{gh}$ we then have $[V,x] = [\Gamma^\perp + Wgh, x] = [Wgh, x] \leq Wgh \cap U = 0$. This shows that $F \cap X^{gh}$ is trivial, in contradiction to confinedness of X. ∎

COROLLARY 5.2: *Let V be the natural module of the classical finitary linear group G, and suppose that X is a confined subgroup in G. Then there exist a unique minimal X-invariant subspace W of finite codimension in V, and a unique maximal X-invariant subspace W_0 in W. In the case when G is a group of isometries of a non-degenerate form on V, the subspace W_0 coincides with the radical $W \cap W^\perp$ of W.*

Proof: It suffices to prove that $W_0 = W \cap W^\perp$ in the case when G is a group of isometries. Clearly, $W \cap W^\perp$ is contained in W_0. On the other hand, $W_0^\perp$ has finite codimension in V, and so $W \leq W_0^\perp$. But then $W_0 = W_0^{\perp\perp} \leq W^\perp$, so that $W_0 \leq W \cap W^\perp$. ∎

As a byproduct we note that there are many confined subgroups in classical finitary linear groups defined over finite fields.

COROLLARY 5.3: *Suppose that the natural module V of the classical finitary linear group G has cardinality $\aleph$ and is defined over a finite field. Then G has $2^\aleph$ confined subgroups.*

Proof: Clearly there are $2^\aleph$ subspaces U_α $(\alpha < 2^\aleph)$ of finite codimension in V. By Propositions 4.6 resp. 4.8, each $X_\alpha = N_G(U_\alpha)$ is confined in G. From Proposition 5.1, there is a unique minimal X_α-invariant subspace W_α of finite codimension in V. Since $W_\alpha \subseteq U_\alpha$, and since V/W_α has just finitely many subspaces for fixed α, we conclude that $|\{W_\alpha \mid \alpha < 2^\aleph\}| = 2^\aleph$. Hence we also get $2^\aleph$ distinct groups X_α. ∎

We now focus our attention on the action of X on its unique infinite-dimensional composition factor W/W_0 in V.

LEMMA 5.4: *In the situation of Corollary 5.2, let $C = C_G(V/W) \cap C_G(W_0)$, and let $R = C_C(W/W_0)$ be the stabilizer in G of the series $V \geq W > W_0 \geq 0$. Then $(X \cap C)R/R$ is a confined subgroup in C/R.*

Proof: Let X be confined with respect to the finite subgroup F of G. Choose a finite-dimensional non-degenerate subspace U in V such that $[V,F] \leq U$ and

$V = U + C_V(F)$. There exists $g \in G$ such that $Ug \leq W$ and $Ug \cap W_0 = 0$. Now $F^g \leq C$ and $F^g \cap R = 1$. Hence $(X \cap C)R/R$ is confined in C/R with respect to $F^g R/R$. ∎

We shall see later in Lemma 7.3, that the derived subgroup of C/R acts on W/W_0 as a classical finitary linear group. Therefore we further examine irreducible confined subgroups.

LEMMA 5.5: *Let b be a non-degenerate symplectic, unitary or orthogonal form on the vector space V. Then $C_V(g) = [V,g]^\perp$ for any finitary isometry g of b.*

Proof: Consider $w \in C_V(g)$. For every $v \in V$ we have $b([v,g],w) = b(vg,w) - b(v,w) = b(vg,wg) - b(v,w) = 0$. This shows that $C_V(g) \leq [V,g]^\perp$. The assertion now follows, since $\dim_{\mathbb{F}} V/[V,g]^\perp = \dim_{\mathbb{F}}[V,g] = \dim_{\mathbb{F}} V/C_V(g)$. ∎

PROPOSITION 5.6: *Let G be a classical finitary linear group, defined over the field $\mathbb{F}$, and let V denote the natural $\mathbb{F}G$-module. Then every irreducible confined subgroup of G acts primitively on V.*

Proof: Let X be confined with respect to the finite subgroup F of G. Assume first that X is totally imprimitive. Then the finite-dimensional subspace $[V,F]$ is contained in a proper X-block in V. Hence, for every $g \in G$, the intersection $F^g \cap X$ normalizes the blocks in the corresponding system Σ of imprimitivity. In particular, the normalizer of Σ is confined in G with respect to F. But this contradicts Proposition 5.1.

Assume next that X is almost-primitive. Then V admits a proper system $\Sigma = \{V_\sigma \mid \sigma \in \Sigma\}$ of imprimitivity under the action of X, and X acts as the alternating group or as the full group of finitary permutations on Σ. Let σ_i ($i \in \mathbb{N}$) be pairwise distinct elements in Σ.

Suppose first that G is a special transvection group. Let $\{u_1, \ldots, u_n\}$ be a basis of $U = [V,F]$. For each i we choose $0 \neq u_{ij} \in V_{\sigma_{3i+j}}$ ($j = 0,1,2$). There exists $g \in G$ with $u_i g = v_i = u_{i0} + u_{i1} + u_{i2}$ for all i. It follows that $Ug \cap (V_\sigma + V_\tau) = 0$ for all $\sigma, \tau \in \Sigma$. Consider any $x \in F^g \cap X$. For every σ we have $[V_\sigma, x] \leq Ug \cap (V_\sigma + V_\sigma x) = 0$, and so x must be trivial, in contradiction to confinedness of X with respect to F.

Suppose now that G is a classical finitary linear group of isometries of a non-degenerate form b on V. In this case Lemma 4.2 ensures that $[V,F]$ is contained in a finite-dimensional subspace U of V which is an orthogonal sum of hyperbolic planes. Since V is an irreducible $\mathbb{F}X$-module, it can in particular not be a permutation module under the action of X. Hence there exists a non-trivial finite subgroup H of X which normalizes V_{σ_1} and acts irreducibly on V_{σ_1}. Since X acts highly transitively on Σ, we may assume that H fixes $V_{\sigma_2} \oplus V_{\sigma_3} \oplus V_{\sigma_4}$ pointwise. Let $x \in X$ induce the permutation $(\sigma_1\sigma_2)(\sigma_3\sigma_4)$ on Σ. Then $K = [H, x]$ is contained in X, and Lemma 5.5 gives

$$\bigoplus_{\sigma \notin \{\sigma_1, \sigma_2\}} V_\sigma = C_V(K) \leq [V, K]^\perp = (V_{\sigma_1} \oplus V_{\sigma_2})^\perp.$$

Since b is non-degenerate, the codimension of $(V_{\sigma_1} \oplus V_{\sigma_2})^\perp$ in V equals $\dim_{\mathbb{F}}(V_{\sigma_1} \oplus V_{\sigma_2})$. Therefore $(V_{\sigma_1} \oplus V_{\sigma_2})^\perp = \oplus_{\sigma \notin \{\sigma_1, \sigma_2\}} V_\sigma$, whence $V_{\sigma_1} \oplus V_{\sigma_2}$ is non-degenerate. From high transitivity of G, we see that $V_{\sigma_1} \oplus V_{\sigma_3}$ is non-degenerate too. But V_{σ_3} is orthogonal to V_{σ_1}. Hence V_{σ_1} and then every V_σ is non-degenerate. And V_σ is orthogonal to V_τ for any two distinct elements $\sigma, \tau \in V$.

Let $p = \operatorname{char} \mathbb{F}$. If b is unitary, or if b is orthogonal and $p > 2$, then we can choose non-isotropic vectors $u_{ij} \in V_{\sigma_{(p+1)i+j}}$, and define $v_i = u_{i0} + \cdots + u_{ip}$. Then the subspace $T = \langle v_i \mid i = 1, \ldots, n+2 \rangle$ is non-degenerate and contains an isometric copy of U. In particular there exists $g \in G$ such that $Ug \leq T$, and we may argue as in the special transvection group case. Moreover, if b is symplectic, then every $V_{\sigma_{(p+1)i+j}}$ contains a hyperbolic pair s_{ij}, t_{ij}, and U has a basis consisting of k pairwise orthogonal hyperbolic pairs. We then consider the subspace T generated by the pairwise orthogonal hyperbolic pairs $s_i = s_{i0} + \cdots + s_{ip}$, $t_i = t_{i0} + \cdots + t_{ip}$, and argue as above. Finally, let b be orthogonal and $p = 2$. Put $V_k = V_{\sigma_{3k}} \oplus V_{\sigma_{3k+1}} \oplus V_{\sigma_{3k+2}}$ for every k. Then V_k contains a hyperbolic pair v_k, w_k (see [27], pp. 138–139), which we can use as in the symplectic case in place of the hyperbolic pair in a single block V_{σ_k}.

Since the irreducible finitary linear group X is neither totally imprimitive nor almost primitive, it must be primitive. ∎

Lemma 5.7: *Let V be an infinite-dimensional vector space over the field $\mathbb{K}$, and let $\mathbb{F}$ be a finite proper subfield of $\mathbb{K}$. Suppose that b is a non-degenerate symplectic, unitary or orthogonal form on the $\mathbb{F}$-space V. For every $n \in \mathbb{N}$ there exists a non-degenerate $\mathbb{F}$-subspace U_n of dimension $2n$ in V which does not contain any non-zero $\mathbb{K}$-subspace.*

Proof: We shall construct U_n recursively from $U_0 = 0$. Suppose then that U_{n-1} has been found for some n. Because n-dimensional $\mathbb{F}$-spaces are finite, we may assume without loss that $\mathbb{K}$ is finite. Let $v \in (\mathbb{K}U_{n-1})^\perp$ be a non-zero vector, which is chosen non-isotropic in the unitary or orthogonal case. Assume that the $\mathbb{F}$-space W generated by U_{n-1} and v contains a non-zero $\mathbb{K}$-subspace. Then there exists $u \in U_{n-1}$ such that $\lambda(u+v) \in W$ for every $\lambda \in \mathbb{K}$. Because $\mathbb{F} < \mathbb{K}$, there exists $0 \neq \lambda \in \mathbb{K}$ such that $\lambda(u+v) \in U_{n-1}$. Thus $u+v \in \mathbb{K}U_{n-1}$, whence $v \in \mathbb{K}U_{n-1}$ too, a contradiction. Hence W does not contain a non-zero $\mathbb{K}$-subspace.

Since $(\mathbb{K}U_{n-1})^{\perp} \setminus (\mathbb{K}v)^{\perp}$ is infinite while $\mathbb{K}W$ is finite, there exists $z \in ((\mathbb{K}U_{n-1})^{\perp} \setminus (\mathbb{K}v)^{\perp}) \setminus \mathbb{K}W$. We let U_n be the $\mathbb{F}$-space generated by W and z. As before, U_n does not contain any non-zero $\mathbb{K}$-subspace. Moreover, $\mathbb{F}v + \mathbb{F}z$ is non-degenerate and contained in $(\mathbb{K}U_{n-1})^{\perp}$. Therefore U_n is non-degenerate as the orthogonal sum of U_{n-1} and $\mathbb{F}v + \mathbb{F}z$. ∎

COROLLARY 5.8: *Let the classical finitary linear group G have its natural module defined over the finite field $\mathbb{F}$, and assume that X is a primitive confined subgroup in G. Then $C_{\mathrm{End}(V)}(X) = \mathbb{F} \cdot \mathrm{id}_V$.*

Proof: Let V be the natural $\mathbb{F}G$-module, and let X be confined with respect to the finite subgroup F of G. Assume that $C = C_{\mathrm{End}(V)}(X) > \mathbb{F} \cdot \mathrm{id}_V$. From Schur's Lemma, C is a skew field, and so it contains a field $\mathbb{K} > \mathbb{F}$. We can consider V also as a vector space over $\mathbb{K}$. Let U be a finite-dimensional subspace of V containing $[V, F]$; here we choose U non-degenerate in the case when G is not a special transvection group. From Lemma 5.7 there exists a (non-degenerate) $\mathbb{F}$-subspace W in V with $\dim_{\mathbb{F}} W = \dim_{\mathbb{F}} U$, which does not contain any non-zero $\mathbb{K}$-subspace. Now $Ug = W$ for suitable $g \in G$. Consider some $x \in F^g \cap X$. Since x is $\mathbb{K}$-linear, $[V, x]$ is a $\mathbb{K}$-subspace, contained in $[V, F^g] \leq Ug = W$. It follows that $[V, x] = 0$. And this shows that $F^g \cap X$ is trivial, in contradiction to confinedness of X. ∎

PROPOSITION 5.9: *Let G be a classical finitary linear group over the finite field $\mathbb{F}$, and let V be the natural $\mathbb{F}G$-module. Suppose that X is a primitive confined subgroup in G. Then the commutator subgroup X' is a classical finitary linear group over a subfield $\mathbb{K}$ of $\mathbb{F}$, and $V = \mathbb{F} \otimes_{\mathbb{K}} W$, where W denotes the natural $\mathbb{K}X$-module (or the conatural module in the case when X' is a special transvection group). Moreover $X \leq \mathrm{Aut}(X')$.*

Proof: From [18], Theorem B, the derived subgroup X' is simple, and $X \leq \mathrm{Aut}(X')$. Furthermore, X' is either an alternating group or a classical finitary linear group, by J. Hall's classification [10], Theorem 1.3.

Assume that $\mathrm{Alt}(\Sigma) \leq X \leq \mathrm{FSym}(\Sigma)$ for some infinite set Σ. Then V is the natural $\mathbb{F}X$-module from [2], Theorem B or [9], Theorem 8.2, that is, V has an $\mathbb{F}$-basis of the form $\mathcal{B} = \{v_\sigma \mid \sigma \in \Sigma'\}$, where $v_\sigma = \sigma - \sigma_0$ and $\Sigma' = \Sigma \smallsetminus \{\sigma_0\}$ for a fixed $\sigma_0 \in \Sigma$. For convenience, we choose pairwise distinct elements $\sigma_i \in \Sigma'$ ($i \in \mathbb{N}$).

Suppose that G is a classical finitary linear group of isometries of a non-degenerate form b on V. Since X acts highly transitively on Σ, there exist $c, d \in \mathbb{F}$ such that $b(v_\sigma, v_\tau) = c$ for all $\sigma \neq \tau$, and $b(v_\sigma, v_\sigma) = d$ for all σ. Consider $x = (\sigma_0\sigma_1)(\sigma_2\sigma_3) \in X$. Clearly, $c = b(v_{\sigma_1}x, v_{\sigma_4}x) = b(-v_{\sigma_1}, v_{\sigma_4} - v_{\sigma_1}) = -c + d$, and so $d = 2c$. Because the form b is non-degenerate, it is clear that $c \neq 0$.

We can show now that $V_n = \langle v_{\sigma_1}, \ldots, v_{\sigma_n} \rangle$ is non-degenerate for $n \not\equiv -1 \bmod p$, where $p = \mathrm{char}\,\mathbb{F}$: Assume that $w = \sum_{i=1}^n \lambda_i v_{\sigma_i}$ is a non-zero vector in the radical of V_n for certain $\lambda_i \in \mathbb{F}$. Then $0 = b(w, v_{\sigma_j}) = c \cdot (\lambda_j + \sum_i \lambda_i)$ for every j, whence $\lambda_i = \lambda_j$ for all i, j. We may thus assume that $w = \sum_{i=1}^n v_{\sigma_i}$. But now $0 = (w, v_{\sigma_1}) = n + 1$, in contradiction to our choice of n.

Let X be confined with respect to the finite subgroup F in G. After conjugation with some element from X we may assume that $[V, F] \leq V_n$ for some $n \not\equiv -1 \bmod p$. Let $w_i = v_{\sigma_{n+2(i-1)p+1}} + \cdots + v_{\sigma_{n+2ip}}$ for $1 \leq i \leq n$. Straightforward calculations show that each w_i is isotropic resp. singular, and perpendicular to each v_σ which does not occur in the expansion of w_i with respect to the basis $\mathcal{B}$. Therefore the assignment $v_i \longmapsto v_i + w_i$ defines an isometry $V_n \longrightarrow \langle v_{\sigma_1} + w_1, \ldots, v_{\sigma_n} + w_n \rangle$, which is induced from some $g \in G$. It follows that $Ug \cap \langle v_\sigma, v_\tau, v_\mu \rangle = 0$ for all $\sigma, \tau, \mu \in \Sigma'$. Consider any $x \in F^g \cap X$. For every $\sigma \in \Sigma'$ we have $[v_\sigma, x] \in Ug \cap \langle v_\sigma, v_{\sigma x}, v_{\sigma_0 x} \rangle = 0$. This shows that $F^g \cap X$ must be trivial, in contradiction to confinedness of X with respect to F.

From the above contradiction, G must be a special transvection group. Again we may assume that $[V, F] \leq V_n$ for some n. But now there exists $g \in G$ with $u_i g = v_{\sigma_{4i}} + \cdots + v_{\sigma_{4i+3}}$ for $1 \leq i \leq n$. And so we reach the same kind of contradiction as before.

We have now proved that X' is a classical finitary linear group, defined over some field $\mathbb{K}$. In the case when X' is a group of isometries of a non-degenerate form, the remaining assertions of the proposition will be deduced from applying [22], Theorem A. In the notation of this theorem, we let $K = \mathbb{K}$ and $R = \mathbb{F}$. Thus M is just an $\mathbb{F}$-vector space. If its $\mathbb{F}$-dimension is d, then $M \otimes_{\mathbb{K}} N$ is the direct sum of d irreducible $\mathbb{F}X'$-submodules, whence $d = 1$. Since M is also a $\mathbb{K}$-module, we have $\mathbb{K} \leq \mathbb{F}$, and $V = \mathbb{F} \otimes_{\mathbb{K}} W$ for the natural $\mathbb{K}X'$-module W. In the case when X' is a special transvection group, we similarly apply [22], Theorem

B. ∎

In the sequel we shall frequently consider elements in classical finite groups G as matrices. Suppose that the finite-dimensional vector space V with non-degenerate symplectic, unitary or orthogonal form κ is an orthogonal sum of hyperbolic planes. There exists a decomposition $V = V_1 \oplus V_2$ into two totally isotropic subspaces V_i. If a basis of V is the union of suitable bases of V_1 and of V_2, then the form κ is given by its Gram matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

And every isometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of κ satisfies

$$\begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^tC \pm C^tA & A^tD \pm C^tB \\ B^tC \pm D^tA & B^tD \pm D^tB \end{pmatrix}$$

where the superscript t denotes transposition when the form is symplectic or orthogonal, and where t denotes transposition and application of the involutory field automorphism when the form is unitary. This equality holds if and only if

(1) $A^tD \pm C^tB = I$, and

(2) A^tC and B^tD are symmetric in the symplectic case, resp. antisymmetric in the unitary or orthogonal case.

In particular, if V_1 and V_2 have even dimension, then for any λ, μ from the underlying field, we may choose

$$A = \begin{pmatrix} I & \mu I \\ \lambda I & (\lambda\mu + 1)I \end{pmatrix}, \quad D = \begin{pmatrix} (\lambda\mu + 1)I & -\mu I \\ -\lambda I & I \end{pmatrix}^t, \quad \text{and} \quad B = C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that in the orthogonal case, this choice is also compatible with the quadratic form from which κ is derived, provided that the basis consists of singular vectors.

Consider a vector space $\widetilde{V}$ isometric to V. Let g act on V and on $\widetilde{V}$ as described by the above choice of matrices. Then g is a commutator in the isometry group of $V \oplus \widetilde{V}$, and hence an element in the corresponding classical group.

LEMMA 5.10: *Let W and V be infinite-dimensional vector spaces over fields $\mathbb{K} \leq \mathbb{F}$ (resp.), and let κ resp. b be non-degenerate symplectic, unitary, or orthogonal forms on W resp. V. Let X resp. G be the corresponding classical finitary linear groups. Suppose that $X \leq G$ and $V = \mathbb{F} \otimes_{\mathbb{K}} W$. Then every isotropic vector in (W, κ) (resp. singular vector when the form κ is orthogonal) is isotropic in (V, b) (resp. singular when the form b is orthogonal), and there exists $\delta \in \mathbb{F}$ such that $b(u, v) = \delta$ for every hyperbolic pair u, v in (W, κ).*

Proof: We prove first that every hyperbolic pair u_0, v_0 in (W, κ) generates a hyperbolic plane in (V, b). Let U_0 be the $\mathbb{K}$-span of u_0, v_0. For $i = 1, \ldots, 6$, consider a hyperbolic pair u_i, v_i in $(U_0 \oplus \ldots \oplus U_{i-1})^{\perp}$ with respect to κ, generating the $\mathbb{K}$-space U_i. Then there exists an element $x \in X$, which fixes the orthogonal of $U_0 \oplus \ldots \oplus U_3$ in (W, κ) pointwise, and whose action on $U_0 \oplus U_1$ resp. on $U_2 \oplus U_3$ is described by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

resp. its inverse, relative to the basis $\{u_0, u_1, v_0, v_1\}$ resp. $\{u_2, u_3, v_2, v_3\}$. The eigenvalues of this matrix are different from 1, and hence

$$[V, x] = \mathbb{F} \otimes_{\mathbb{K}} (U_0 \oplus \cdots \oplus U_3).$$

It now follows from Lemma 5.5 that $[V, x]$ is non-degenerate with respect to b too, and that $[V, x]^{\perp} = C_V(x)$ is independent of the form.

As above there exists an element $y \in X$, such that

$$[V, y] = \mathbb{F} \otimes_{\mathbb{K}} (U_0 \oplus U_4 \oplus U_5 \oplus U_6) \quad \text{and} \quad [V, y]^{\perp} = C_V(y).$$

Also $[V, y]$ is non-degenerate with respect to both forms, and $[V, y]^{\perp} = C_V(y)$ is independent of the form. However,

$$(\mathbb{F} \otimes_{\mathbb{K}} U_0)^{\perp} = ([V, x] \cap [V, y])^{\perp} = [V, x]^{\perp} + [V, y]^{\perp}$$

is then independent of the form too. In particular, $\overline{U}_0 = \mathbb{F} \otimes_{\mathbb{K}} U_0$ is non-degenerate in (V, b).

We shall prove next that the hyperbolic pair u_0, v_0 in (W, κ) consists of vectors which are isotropic resp. singular in (V, b). In the case when the characteristic is 2 and b is the associated orthogonal form of a quadratic form q on

V, we consider the above element $x \in X \leq G$. Obviously, $q(u_0) = q(u_0 x) = q(u_0 + u_1) = q(u_0) + q(u_1) + b(u_0, u_1)$. Hence $0 = q(u_1) = q(u_1 x) = q(u_0)$. Similarly, $q(v_0) = 0 = q(v_1)$. In all other cases, we consider the transformation $z \in X$, which fixes the orthogonal of $U_0 \oplus \cdots \oplus U_3$ in (W, κ) pointwise, and whose action on $U_0 \oplus \cdots \oplus U_3$ is described by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

relative to the basis $\{u_0, u_1, v_0, v_1\}$ resp. $\{u_2, u_3, v_2, v_3\}$. Then

$$u_0 \in [V, z] \cap C_V(z) = [V, z] \cap [V, z]^{\perp}$$

by Lemma 5.5, and so u_0 is isotropic with respect to both forms. This applies to any isotropic vector in (W, κ). By Witt's Theorem (see [27], Theorem 7.4), X acts transitively on the set of hyperbolic pairs in (W, κ), and this furnishes the existence of δ. ∎

PROPOSITION 5.11: *In the situation of Proposition 5.9 we have* $\mathbb{K} = \mathbb{F}$.

Proof: Assume that there exists $\mu \in \mathbb{F} \setminus \mathbb{K}$. Note that X acts on V by matrices with entries from $\mathbb{K}$, relative to a $\mathbb{K}$-basis B of W. Let X be confined with respect to the finite subgroup F of G.

We first consider the case when $X' = \mathrm{T}_{\mathbb{K}}(\Gamma, W)$ is a special transvection group. Assume that G is a classical finitary linear group of isometries of a non-degenerate form b on V. As in the proof of Lemma 5.10 there exists $x \in X'$ such that $\dim_{\mathbb{K}}[W, x] = 4$ and $W = [W, x] \oplus C_W(x)$. Let $U = [W, x]$. Lemma 5.5 gives that $\overline{U} = [V, x]$ is non-degenerate with respect to b. From $\overline{U} = \mathbb{F} \otimes_{\mathbb{K}} U$ we obtain that $b|_{U \times U} \neq 0$. However, $N_{X'}(U)$ acts on U as a full general linear group, and in particular transitively on the set of pairs of $\mathbb{K}$-independent vectors. Let $u, v, w \in U$ be $\mathbb{K}$-independent. Clearly $b(u, v) = b(u, u+v) = b(u, u) + b(u, v)$, and so every vector in U is isotropic. Moreover, $b(u, v) = b(u, v+w) = b(u, v) + b(u, w)$ yields $b(u, w) = 0$. Altogether this contradicts $b|_{U \times U} \neq 0$.

Hence G is a special transvection group $\mathrm{T}_{\mathbb{F}}(\Delta, V)$ too. We may assume that $[V, F] \leq \overline{U}_1$ and $V = \overline{U}_1 + C_V(F)$. Let $\overline{U}_2$ be a further subspace of V, with $\dim_{\mathbb{F}} \overline{U}_1 = \dim_{\mathbb{F}} \overline{U}_2$ and $\overline{U}_1 \cap \overline{U}_2 = 0$, such that $\overline{U}_1$ and $\overline{U}_2$ are generated by subsets B_1 and B_2 of B. Then there exist finite-dimensional subspaces $\Delta_0 \leq \Delta$, and $\overline{U} \leq V$ containing $\overline{U}_1 \oplus \overline{U}_2$, such that $\mathrm{ann}_{\Delta_0} \overline{U} = 0$ and $\mathrm{ann}_{\overline{U}} \Delta_0 = 0$.

Now $\Delta_0^\perp$ is a complement to $\overline{U}$ in V such that $N_G(\overline{U})\cap C_G(\Delta_0^\perp)$ induces the full $\mathrm{SL}_{\mathbb{F}}(\overline{U})$ on $\overline{U}$. Choose $u_i \in \overline{U}_i$, and let $B_1 \cup B_2 = \{u_1, u_2\} \dot\cup B_0$. There exists $g \in N_G(\overline{U}_1 \oplus \overline{U}_2)$, which fixes B_0 and $\Delta_0^\perp$ pointwise, and whose action on $\mathbb{F}u_1 \oplus \mathbb{F}u_2$ is described by the matrix

$$\begin{pmatrix} 1 & \mu \\ 1 & \mu+1 \end{pmatrix}.$$

Since X is confined, there exists a non-trivial element $x \in F \cap X^g$. It follows that $0 \neq [V,x] = [Bg,x] \leq [V,F] \cap [Bg, X^g] \leq \overline{U}_1 \cap \mathbb{K}Bg$. So we consider $0 \neq w \in \overline{U}_1 \cap \mathbb{K}Bg$. There must exist $a, b \in \mathbb{K}$ such that

$$w \in au_1g + bu_2g + \mathbb{K}B_0 = a(u_1+u_2) + b(\mu u_1 + (\mu+1)u_2) + \mathbb{K}B_0.$$

Comparing coefficients at u_2 we see that $a + b(\mu+1) = 0$, whence $a = 0 = b$, a contradiction. This contradiction shows that the assertion of the proposition holds when X' is a special transvection group.

Suppose next that X' is a classical finitary linear group of isometries of a non-degenerate form κ on W. If G is a special transvection group, we can argue as above. Hence G is a classical finitary linear group of isometries of a non-degenerate form b on V. Let U_1 be an orthogonal sum of hyperbolic planes in (W,κ) such that $[V,F] \leq \overline{U}_1 = \mathbb{F} \otimes_{\mathbb{K}} U_1$ and $V = \overline{U}_1 + C_V(F)$ (Lemma 4.2). Let $U_2 \leq U_1^\perp$ be isometric to U_1 with respect to κ, and let $U = U_1 \oplus U_2$. We may assume that B is the union of a $\mathbb{K}$-basis of $U^\perp$ with $\mathbb{K}$-bases of U_1 and U_2 consisting of pairwise orthogonal hyperbolic pairs. Let u_i, v_i be the first hyperbolic pair in the basis of U_i, and let $B_0 = B \setminus \{u_1, u_2, v_1, v_2\}$. Let $\overline{U}$ be the $\mathbb{F}$-subspace of V spanned by u_1, u_2, v_1, v_2. Due to Lemma 5.10, the forms $\kappa|_{\overline{U}}$ and $b|_{\overline{U}}$ have Gram matrices

$$\begin{pmatrix} 0 & I \\ \pm I & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & \delta I \\ \pm(\delta I)^t & 0 \end{pmatrix}$$

relative to $\{u_1, u_2, v_1, v_2\}$, and in the case when b is orthogonal, the vectors u_1, u_2, v_1, v_2 are singular in (V, b). Hence there exists $g \in N_G(\overline{U}) \cap N_G(\overline{U}^\perp)$, whose action on $\overline{U}$ is described by the matrix

$$\begin{pmatrix} 1 & \mu & 0 & 0 \\ 1 & \mu+1 & 0 & 0 \\ 0 & 0 & \mu^t+1 & -1 \\ 0 & 0 & -\mu^t & 1 \end{pmatrix}.$$

Since X is confined, there exists a non-trivial element $x \in F \cap X^g$. It follows that $0 \neq [V,x] = [Bg,x] \leq [V,F] \cap [Bg, X^g] \leq \overline{U}_1 \cap \mathbb{K}Bg$. So we consider

$0 \neq w \in \overline{U}_1 \cap \mathbb{K}Bg$. There must exist $a, b, c, d \in \mathbb{K}$ such that

$$\begin{aligned} w &\in au_1g + bu_2g + cv_1g + dv_2g + \mathbb{K}B_0 \\ &= a(u_1 + u_2) + b(\mu u_1 + (\mu + 1)u_2) \\ &\quad + c((\mu^t + 1)v_1 - \mu^t v_2) + d(v_2 - v_1) + \mathbb{K}B_0. \end{aligned}$$

Comparing coefficients at u_2 and v_2, we see that $a + b(\mu + 1) = 0$ and $d - c\mu^t = 0$, whence the coefficients a, b, c, d must all be zero, a contradiction. ∎

PROPOSITION 5.12: *Let G be a classical finitary linear group over the finite field $\mathbb{F}$, and let V be the natural $\mathbb{F}G$-module. Suppose that X is a proper primitive confined subgroup in G. Then one of the following two alternatives holds.*

(1) *X and G are special transvection groups, and V is the natural or the conatural module for X, or*
(2) $\operatorname{char} \mathbb{F} = 2$ *and* $X' = \mathrm{F}\Omega_{\mathbb{F}}(V) \leq \mathrm{FSp}_{\mathbb{F}}(V) = G$.

Proof: Suppose that X is confined with respect to the finite subgroup F of G. From Proposition 5.11 we know that X' is a classical finitary linear group over $\mathbb{F}$, and that V is the natural $\mathbb{F}X'$-module (or the conatural module in the case when X' is a special transvection group). If X' is a special transvection group, then it follows as in the proof of Proposition 5.11 that G is a special transvection group. In particular, every element in G has determinant 1. Thus X is perfect, and we are in case (1).

Suppose next that X' is a classical finitary linear group of isometries of a non-degenerate form κ on V. If $x \in X$, then V has a local system of non-degenerate subspaces U in V satisfying $[V, x] \leq U$ and $V = U + C_V(x)$. Lemma 5.5 yields that, for every such U, the element x induces a transformation of U which normalizes the finite classical group induced by $N_{X'}(U) \cap C_{X'}(U^\perp)$ on U. Hence x is an isometry of (V, κ) too.

Assume that G is a special transvection group. Let W be a sum of hyperbolic planes with respect to κ, which contains $[V, F]$ and supplements $C_V(F)$. Let $T_1 \oplus T_2$ be a non-degenerate subspace of V which is the orthogonal sum of two totally isotropic subspaces T_i in V of the same dimension as W. Since V has a local system of subspaces U with the property that $N_G(U)$ induces the full special linear group on U, there exists $g \in G$ with $Wg = T_1$. Consider a non-trivial $x \in F^g \cap X$. On the one hand we can find $t_i \in T_i$ such that $\kappa(t_1, t_2) \neq 0$. On the other hand $[V, x] \leq T_1$ implies $\kappa(t_1, t_2) = \kappa(t_1x, t_2x) = \kappa(t_1x, [t_2, x]) = 0$. This contradiction shows that G too must be a classical finitary linear group of isometries of a non-degenerate form b on V.

Let $p = \operatorname{char}\mathbb{F}$. From Lemma 5.10, the two forms match quite well on finite-dimensional subspaces which are orthogonal sums of hyperbolic planes. Some possibilities can therefore be ruled out by comparing the p-parts of the orders of finite classical groups acting on such subspaces — since the normalizers in classical finitary linear groups of these subspaces are finite classical groups. Namely, $|\mathrm{Sp}(4,\mathbb{F})|_p > |\mathrm{SU}(4,\mathbb{F})|_p > |\Omega^+(4,\mathbb{F})|_p$. Therefore it just remains to rule out the following cases:

(i) κ unitary and b symplectic;

(ii) κ orthogonal and b unitary;

(iii) κ orthogonal, b symplectic, and $\operatorname{char}\mathbb{F} \neq 2$.

In case (i) we consider the unitary, but not symplectic 4×4-matrix

$$\begin{pmatrix} 1 & \mu & 0 & 0 \\ 1 & \mu+1 & 0 & 0 \\ 0 & 0 & \mu^t+1 & -1 \\ 0 & 0 & -\mu^t & 1 \end{pmatrix} \quad \text{with} \quad \mu^t \neq \mu.$$

In case (ii) we consider the orthogonal, but not unitary 4×4-matrix

$$\begin{pmatrix} 1 & \mu & 0 & 0 \\ 1 & \mu+1 & 0 & 0 \\ 0 & 0 & \mu+1 & -1 \\ 0 & 0 & -\mu & 1 \end{pmatrix} \quad \text{with} \quad \mu^t \neq \mu.$$

In case (iii) we consider the 4×4-matrix

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \quad \text{with} \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Because of $\operatorname{char}\mathbb{F} \neq 2$, this matrix is orthogonal, but not symplectic. ∎

6. Cohomological considerations

We shall now derive a result about the cohomology of finitary linear groups which will be needed in our classification of confined subgroups. For the relevant definitions and basic facts about group cohomology, we refer the reader to [14].

LEMMA 6.1: *Let G be the special linear group acting on the n-dimensional vector space V. The dual space V^* is isomorphic to the exterior power $\bigwedge^{n-1} V$ as a G-module.*

Proof: Consider the isomorphism φ which takes every $\alpha \in \bigwedge^{n-1} V$ to $\varphi_\alpha \in \mathrm{Hom}_{\mathbb{F}}(V,\mathbb{F})$, where φ_α is defined via $v\varphi_\alpha = v \wedge \alpha \in \bigwedge^n V \cong \mathbb{F}$ for all $v \in V$.

Since G has no non-trivial 1-dimensional representation, it acts trivially on $\bigwedge^n V$. Hence $(v)\varphi_\alpha g = (vg^{-1})\varphi_\alpha = vg^{-1} \wedge \alpha = v \wedge \alpha g = v\varphi_{\alpha g}$ for all $v \in V$, $g \in G$, $\alpha \in \bigwedge^{n-1} V$. ∎

LEMMA 6.2: *Let G_n ($n \in \mathbb{N}$) be a family of finite classical groups of fixed type over a fixed finite field $\mathbb{F}$, such that the natural G_n-module V_n is an orthogonal sum of n hyperbolic planes in the isometry case. Suppose further that the G_n are not symplectic in the case when* $\operatorname{char}\mathbb{F} = 2$. *Then* $\mathrm{H}^1(G_n, V_n^*) = 0$ *for infinitely many values of n.*

Proof: If $\mathbb{F}$ has odd characteristic, then G_n contains the non-trivial central element, which acts fixed-point-freely on V_n^* as scalar multiplication by -1. Therefore the assertion holds for every n.

Suppose now that $\operatorname{char}\mathbb{F} = 2$. In the case when the G_n are unitary groups and $\mathbb{F} = \mathrm{GF}(2^{2k})$, we choose the n in such a way that they are multiples of $2^{2k} - 1$. Then each G_n contains a central element acting fixed-point-freely on V_n as scalar multiplication by a, where $a \in \mathbb{F} \setminus \mathrm{GF}(2^k)$ satisfies $a^t a = 1$. And so the above argument applies correspondingly. In the case when G is the stable special linear group and $\mathbb{F} > \mathrm{GF}(2)$, we can use the same procedure as for unitary groups. If however $\mathbb{F} = \mathrm{GF}(2)$, then it follows from Lemma 6.1 and [1], Table I that $\mathrm{H}^1(G_n, V_n^*)$ is trivial for all but finitely many n.

In the case when the G_n are orthogonal, we need a different approach. Let H be a finite orthogonal group of a quadratic form of maximal Witt index over the field $\mathbb{F}$ of characteristic 2. Let U be the natural $\mathbb{F}H$-module, and $S = H'$ the corresponding classical orthogonal group. Suppose that $\dim_{\mathbb{F}} U \geq 8$. Consider the Lyndon-Hochschild-Serre spectral sequence

$$0 \longrightarrow \mathrm{H}^1(H/S, V^S) \longrightarrow \mathrm{H}^1(H, V) \longrightarrow \mathrm{H}^1(S, V)^{H/S} \longrightarrow \mathrm{H}^2(H/S, V^S)$$

(see [14], Theorem VIII.9.5). The first and the last term are zero because S acts fixed-point-freely on V. Moreover $H^1(H, V) = 0$ by Corollary 4.3 of [24]. Therefore $H^1(S, V)^{H/S} = 0$ too. Since the group H/S of order 2 cannot act fixed-point-freely on the elementary-abelian 2-group $H^1(S, V)$, we conclude that $H^1(S, V)$ must be trivial. This observation applies to every G_n in place of S, since the dual G_n-module is isomorphic to the natural G_n-module. ∎

PROPOSITION 6.3: *Let V be a vector space over the finite field $\mathbb{F}$, and suppose that G is a classical finitary linear group, which is not finitary symplectic in the* *case when* $\operatorname{char}\mathbb{F} = 2$. *If there exists a G-invariant finite-dimensional subspace* W *in* V *such that* V/W *is the natural* $\mathbb{F}G$*-module, then* $V = W \oplus [V, G]$.

Proof: Let $\overline{V} = V/W$. For every finite subgroup F_0 of G, and for every finite-dimensional subspace U_0 of V, there exists a subspace U in V containing $W + U_0$, such that $[V, F] \leq U$ and $V = U + C_V(F)$, and such that $\overline{U}$ is an orthogonal sum of finitely many hyperbolic planes in the isometry case. Here the dimension of $\overline{U}$ can be chosen of any large enough size. If G is a group of isometries, choose $F = C_G(\overline{U}^{\perp})$. Otherwise there exists a finite-dimensional subspace $\Delta \leq \overline{V}^*$ such that $\operatorname{ann}_\Delta U = 0$ and $\operatorname{ann}_U \Delta = 0$, and we choose $F = N_G(\overline{U}) \cap C_G(\Delta^{\perp})$. In both cases, F acts as the finite classical group on $\overline{U}$. In particular, $F_0 \leq F$. Moreover, the right choice of $\dim_{\mathbb{F}} \overline{U}$ implies that $\mathrm{H}^1(F, \overline{U}^*) = 0$ (Lemma 6.2). Let $k = \dim_{\mathbb{F}} W$. Now [7], Proposition 10.1.2 yields

$$\begin{aligned}\operatorname{Ext}_F(\overline{U}, W) &\cong \mathrm{H}^1(F, \operatorname{Hom}_{\mathbb{F}}(\overline{U}, W)) \cong \mathrm{H}^1(F, \operatorname{Hom}_{\mathbb{F}}(\overline{U}, \mathbb{F})^k) \\ &\cong \mathrm{H}^1(F, \operatorname{Hom}_{\mathbb{F}}(\overline{U}, \mathbb{F}))^k \cong \mathrm{H}^1(F, \overline{U}^*)^k = 0.\end{aligned}$$

In other words, $U = W \oplus [U, F]$. Since every pair (F_0, U_0) of the above form is contained in a pair (F, U) of the above form, it follows at once that $W = W \oplus [V, G]$. ∎

7. Classification of confined subgroups

In the sequel, we shall consider the following situation. Let V be a vector space over the finite field $\mathbb{F}$, equipped with a non-degenerate symplectic, unitary, or orthogonal form b. Let G be the classical finitary linear group of isometries of V. Suppose that W is a subspace of V, with radical $W_0 = W \cap W^{\perp}$. Note that W can be strictly smaller than $W^{\perp\perp}$ (see [6], Chapter I).

LEMMA 7.1: $C_G(V/W) \leq C_G(W^{\perp}) = C_G(V/W^{\perp\perp})$.

Proof: Consider elements $v \in V$, $w \in W^{\perp}$, and $c \in C_G(V/W)$. Then $b(v, [w, c]) = b(v, wc) - b(v, w) = b(vc^{-1}, w) - b(v, w) = b([v, c^{-1}], w) = 0$. Therefore $[W^{\perp}, c] \leq V^{\perp} = 0$, and $c \in C_G(W^{\perp})$. Since $W^{\perp\perp\perp} = W^{\perp}$, we also get $C_G(V/W^{\perp\perp}) \leq C_G(W^{\perp})$. Conversely, if $v \in V$, $w \in W^{\perp}$, and $c \in C_G(W^{\perp})$, then $b([v, c], w) = b(v, [w, c^{-1}]) = 0$. This implies $[V, c] \leq W^{\perp\perp}$, and $C_G(W^{\perp}) \leq C_G(V/W^{\perp\perp})$. ∎

Consider $C = C_G(V/W)$ and $Z = C_G(V/W_0) \cap C_G(W)$. Clearly, $[V, C, Z] = 0 = [V, Z, C]$, so that $[C, Z] = 0$. In fact, a similar calculation shows that Z is a normal subgroup in $N_G(W)$. The stabilizer R of the series $V \geq W \geq W_0 \geq 0$ is unipotent (see [23], Theorem B) and nilpotent of class at most 2. Note that $R' \leq Z \leq R \leq C$, and that Z is isomorphic to a subgroup of $\mathrm{Hom}(V/W, W_0)$. In particular, if W has finite codimension in V, then Z is finite.

We denote images of elements from $N_G(W)$ modulo Z by bars, and images in V modulo W_0 by $\widetilde{\ }$. The group $\overline{C}$ acts on $\overline{R}$ via conjugation, and on $\Theta = \mathrm{Hom}(\widetilde{W}, W_0)$ via $\varphi^c = c^{-1}\varphi c$ for all $\varphi \in \Theta$, $c \in C$. For every $x \in R$, a homomorphism $\tau_x\colon \widetilde{W} \longrightarrow W_0$ is given by $\widetilde{w}\tau_x = [w, x]$ for all $v \in V$, $w \in W$.

LEMMA 7.2: *$C_G(V/W) \cap C_G(W) \leq Z$. In particular, the map $\tau\colon \overline{R} \ni \overline{x} \longmapsto \tau_x \in \mathrm{Hom}(\widetilde{W}, W_0)$ is a monomomorphism of $\overline{C}$-modules.*

Proof: Straightforward calculations show that τ is a $\overline{C}$-homomorphism. Consider elements $v \in V$, $w \in W$, and $x \in \ker\tau = C_G(V/W) \cap C_G(W)$. Then $b([v,x], w) = b(v, [w, x^{-1}]) = 0$. Therefore $[V, x] \leq W \cap W^{\perp} = W_0$, and $x \in Z$. ∎

We finally note that b induces a non-degenerate form $\widetilde{b}$ on $\widetilde{W}$, and that every element in C induces a transformation on $\widetilde{W}$ which is an isometry with respect to $\widetilde{b}$. Correspondingly, a quadratic form q with associated bilinear form b on V induces a quadratic form $\widetilde{q}$ with associated bilinear form b on $\widetilde{W}$, whenever the q-radical $\mathrm{rad}_q(W_0) = \{w \in W_0 \mid q(w) = 0\}$ coincides with W_0. This is always the case in odd characteristic.

LEMMA 7.3: *Consider the above situation, and suppose in addition that $\mathrm{rad}_q(W_0) = W_0$ when b is the associated bilinear form of a quadratic form q. Then the derived subgroup of C induces the classical finitary linear group on $\widetilde{W}$ with respect to $\widetilde{b}$ resp. $\widetilde{q}$.*

Proof: Let U be a non-degenerate complement to W_0 in W. Via the canonical homomorphism $\widetilde{W} \cong U$, any isometry $\widetilde{\varphi}$ in the classical finitary linear group on $\widetilde{W}$ induces an isometry φ in the classical finitary linear group on U. By Lemma 4.2, there exists a finite-dimensional non-degenerate subspace U_0 in U such that $[U, \varphi] \leq U_0$. Note that $U \cap U_0^{\perp} \leq U \cap [U, \varphi]^{\perp} = C_U(\varphi)$ from Lemma 5.5. Moreover $V = U_0 \oplus U_0^{\perp}$. Hence we may choose $c \in C_G(U_0^{\perp})$ such that c acts on U_0 like φ. Then $c \in C$ induces $\widetilde{\varphi}$ on $\widetilde{W}$. This shows that C induces every isometry from the classical finitary linear group on $\widetilde{W}$. It follows that C' acts like the classical finitary linear group on $\widetilde{W}$. ∎

From now on, let X be a confined subgroup of G, and let W denote the minimal X-invariant subspace of finite codimension in V (see Corollary 5.2). In this situation the assumption in Lemma 7.3 is always fulfilled.

LEMMA 7.4: *Let b be orthogonal with quadratic form q on V. Then $\operatorname{rad}_q(W_0) = W_0$.*

Proof: Assume that $\operatorname{rad}_q(W_0)$ is a proper subspace of W_0. Then it has codimension 1 in W_0. Note that $\operatorname{rad}_q(W_0)$ is X-invariant. For any $w \in W$ and any $x \in R$ we have $0 = q(wx) - q(w) = q([w,x]) + b(w,[w,x]) = q([w,x])$. This shows that R acts trivially on $W/\operatorname{rad}_q(W_0)$. But $X'R/R$ is infinite and simple, and so $W/\operatorname{rad}_q(W_0)$ is an extension of the trivial $X'R/R$-module by the natural $X'R/R$-module. By Proposition 6.3, the extension splits. Hence X normalizes the subspace $[W, X']$ of codimension 1 in W. This contradicts the minimal choice of W. ∎

PROPOSITION 7.5: *If b is orthogonal, then $|C : C \cap XR| \leq 2$. If $\operatorname{char} \mathbb{F}$ is odd or if b is unitary, then $C \leq XR$.*

Proof: Every element in C induces an isometry on $\widetilde{W}$ with determinant 1. Therefore, if b is not orthogonal, then Lemma 7.3 ensures that C induces the classical finitary linear group of isometries of $\widetilde{b}$ on $\widetilde{W}$. It now follows from Propositions 5.6 and 5.12 that $C/R \leq XR/R$.

Suppose next that b is orthogonal. Then C' induces the finitary orthogonal group of isometries of $\widetilde{b}$ on $\widetilde{W}$, while C may induce the group of all finitary isometries of $\widetilde{b}$ with determinant 1. In particular, $C'R$ has index at most 2 in C (see [27], Theorem 11.51). ∎

We are now well-prepared to give the proof of Theorem A.

Proof of part (b) of Theorem A: By Corollary 5.2 there exists a minimal X-invariant subspace W of finite codimension in V, and $W_0 = W \cap W^{\perp}$ is the largest X-invariant subspace of finite dimension in W. We shall now stick to the notation introduced above. It is clear from Proposition 7.5 that the derived subgroup C' and the finite residual C^o of C are contained in XR. We will show in the sequel that $R \leq XZ$. It then follows that $C^o \leq XZ$, whence $|C^o : C^o \cap X| = |C^o X : X| \leq |ZX : X| = |Z : Z \cap X| < \infty$, and $C^o \leq X$.

Fix some $r \in R$. We now reduce our considerations to the countable situation. There exists a non-degenerate subspace W_1 in V such that $W = W_0 \oplus W_1$. By Lemma 4.2, W_0 is contained in a non-degenerate subspace $S \leq V$, which is an orthogonal sum of $\dim_{\mathbb{F}} W_0$ hyperbolic planes. And so $S = W_0 \oplus S_0$ for some totally isotropic subspace S_0 of S. Because $W_1 \leq W_0^{\perp}$, the space S has trivial intersection with W_1. We need to establish $W = (W \cap S) \oplus (W \cap S^{\perp})$: Because $V = S \oplus S^{\perp}$, every $w \in W$ has the form $w = s_1 + s_2$ for suitable $s_1 \in S$, $s_2 \in S^{\perp}$. However, $W_0 \leq S$, and so $S^{\perp} + W \leq W_0^{\perp}$. It follows that $s_1 = w - s_2 \in S \cap W_0^{\perp} = W \cap S$ and $s_2 = w - s_1 \in W \cap S^{\perp}$.

Recall that $D = X \cap C'R$ induces the classical finitary linear group on $\widetilde{W}$ (Proposition 7.5). We define ascending chains of finite-dimensional subspaces $U_n \leq W_1$ ($n \in \mathbb{N}$), and of finite subgroups $X_n \leq D$ ($n \in \mathbb{N}$) as follows. Each U_n shall be the orthogonal sum of U_{n-1} and some additional hyperbolic planes. Choose U_0 and X_0 such that $[V, r] \leq U_0 \oplus W_0$ and $V = U_0 + C_V(r)$, such that $W_0 \leq [U_0, X_0]$, and such that X_0R/R induces the finite classical group $C_{D/R}(\widetilde{U}_0^{\perp})$ on $\widetilde{U}_0$. In the general step $n \to n+1$, we choose U_{n+1} and X_{n+1} such that $[V, X_n] \leq U_{n+1} \oplus W_0$ and $V = U_{n+1} + C_V(X_n)$, and such that $X_{n+1}R/R$ induces the finite classical group $C_{D/R}(\widetilde{U}_{n+1}^{\perp})$ on $\widetilde{U}_{n+1}$.

Consider $W_2 = \bigcup_n U_n$ and $\widehat{X} = \bigcup_n X_n$. Clearly, $V_2 = W_0 \oplus W_2 \oplus S_0$ is non-degenerate. From Lemmata 7.2 and 7.3, $C_G(V/V_2)$ is the classical finitary linear group with natural module V_2. Every finite subgroup F of G has a conjugate in $C_G(V/V_2)$. Therefore $C_X(V/V_2)$ is confined in $C_G(V/V_2)$. By construction, the subgroup $\widehat{X}$ of $C_X(V/V_2)$ acts irreducibly on $\widetilde{W}_2$, and $W_0 \leq [W_2, \widehat{X}]$. Hence $W_0 \oplus W_2 = W \cap V_2$ is a minimal $\widehat{X}$-invariant subspace of finite codimension in V_2. And finally, the stabilizer in $C_G(V/V_2)$ of the series $V_2 \geq W \cap V_2 > W_0 \geq 0$ contains our fixed element $r \in R$. We also need to establish that $Z_2 = C_G(V/V_2) \cap C_G(V_2/W_0) \cap C_G(W \cap V_2)$ is contained in Z: Consider some $z_2 \in Z_2$. The restriction of z_2 to V_2 stabilizes the series $S > W_0 \geq 0$ in S and centralizes $V_2 \cap S^{\perp}$. And so it extends to some $z \in Z$ which centralizes $S^{\perp}$. However, $Z \leq Z_2$. It follows that $z_2 z^{-1} \in C_G(V/V_2) \cap C_G(V_2) = 1$, whence $z_2 \in Z$. All of this shows that we may replace V, G, X, W by V_2, $C_G(V/V_2)$, $C_X(V/V_2)$, $W \cap V_2$ (resp.), and assume in this way that V and G are countable.

Because $\overline{C}$ acts trivially on W_0, the $\overline{C}$-module $\mathrm{Hom}(\widetilde{W}, W_0)$ is isomorphic to a direct sum of finitely many copies of $\mathrm{Hom}(\widetilde{W}, \mathbb{F})$. Namely, $\mathrm{Hom}(\widetilde{W}, W_0) \cong \bigoplus_{i=1}^{k} \mathrm{Hom}(\widetilde{W}, \langle w_i \rangle)$ for a basis $\{w_1, \ldots, w_k\}$ of W_0. For each $\widetilde{w} \in \widetilde{W}$ define the form $\phi_{\widetilde{w}}$ via $(\widetilde{u})\phi_{\widetilde{w}} = b(\widetilde{u}, \widetilde{w})$. Then

$$(\widetilde{u})\phi_{\widetilde{w}}^{c} = (\widetilde{u}c^{-1})\phi_{\widetilde{w}} = b(\widetilde{u}c^{-1}, \widetilde{w}) = b(\widetilde{u}, \widetilde{w}c) = (\widetilde{u})\phi_{\widetilde{w}c} \quad \text{for every } c \in \overline{C}.$$

This shows that the map $\phi\colon \widetilde{w} \longmapsto \phi_{\widetilde{w}}$ is an isomorphism of $\overline{C}$-modules, except in the unitary case, where ϕ is semilinear instead of linear. Hence $M = \langle \phi_{\widetilde{w}} \mid \widetilde{w} \in \widetilde{W} \rangle$ is an irreducible $\overline{D}$-module (see [20], Proposition 7.5 and [19], Section 2.2). Since G is countable, W has a basis with respect to which C acts on W by stable matrices (see [17], Theorem 2.1). For each $x \in R$, the map τ_x annihilates all but finitely many vectors of this standard basis. Therefore the image of τ is isomorphic to a $\overline{D}$-submodule of M^k. It follows from Lemma 7.2 that $\overline{R}$ is a direct sum of copies of M.

There exists a non-degenerate subspace W_1 in V such that $W = W_0 \oplus W_1$. Now $\overline{X \cap R}$ has a $\overline{D}$-invariant complement $\overline{B}$ in $\overline{R}$, and so $\overline{DR}$ is a split extension of $\overline{B}$ by $\overline{D}$. On the other hand, Lemmata 7.2 and 7.3 show that $C_G(V/W_1) \cap DR$ has trivial unipotent radical and induces the same group of transformations on $\widetilde{W}$ as DR. Therefore it is a complement to R in DR. Since $\overline{DR}/\overline{B} \cong \overline{D}$, also $\overline{D}$ splits over $\overline{X \cap R}$, with some complement $\overline{Y}$. Note that Y is the extension of the finite abelian p-group Z by the classical finitary linear group $\overline{Y}$. Because $\overline{Y}$ is infinite and simple, Z is central in Y. We can write $\overline{Y}$ as the union of a local system of finite classical groups. Their Schur multipliers are trivial. Hence $Y = Y' \times Z$, and $Y' \leq (DZ)' \leq D'[D, Z] = D' \leq X$. Because Y' acts as the classical finitary linear group on $\widetilde{W}$, Proposition 6.3 implies that $W = W_0 \oplus [W, Y']$.

Assume that $\overline{X \cap R}$ is a proper $\overline{D}$-submodule of $\overline{R}$. Then $\overline{X \cap R}$ is a direct sum of at most $k-1$ copies of M, and $[W, X \cap R]$ is a proper subspace of W_0. However, $Y'(X \cap R)Z$ normalizes $\widehat{U} = [W, Y'] + [W, X \cap R]$. Let T be a transversal of D in X. Because D has finite index in X, the intersection $\bigcap_{t \in T} \widehat{U}t$ is a proper X-invariant subspace of finite codimension in W. This contradiction to the choice of W shows that $R \leq XZ$, as desired. ∎

Proof of part (a) of Theorem A: Here C acts on $\widetilde{W}$ as the finitary symplectic group with respect to $\widetilde{b}$ (Lemma 7.3), and the only difference is caused by the action of X on $\widetilde{W}$. Namely, Lemma 5.4 and Propositions 5.6/5.12 imply that either X acts on $\widetilde{W}$ like C (so that we can again argue as in part (b) of Theorem A), or that X' acts on $\widetilde{W}$ like a classical finitary orthogonal group with respect to a quadratic form $\widetilde{q}$ with associated bilinear form $\widetilde{b}$. In this latter case, we choose a complement U to W_0 in W, and an $\mathbb{F}$-basis $\{v_i + W \mid i \in I\}$ of V/W. Then a quadratic form q on W with associated bilinear form b is given via $q(v_i) = 0$ for all i, and $q(u + w) = \widetilde{q}(u)$ for all $u \in U$, $w \in W_0$. Note that $W_0 = \mathrm{rad}_q(W_0)$. In particular, R is contained in the finitary orthogonal group H of isometries of q. Therefore, the proof of part (b) can be imitated with H in place of G. ∎

The finite sections in C can in fact be located more precisely.

PROPOSITION 7.6: *Let $p = \operatorname{char} \mathbb{F}$ and $k = \dim_{\mathbb{F}} W_0$.*

(a) *If p is odd, then $R \leq X$. In particular, $C \leq X$ whenever b is symplectic or unitary, while $|C : C \cap X| \leq 2$ when b is orthogonal.*

(b) *If* $p = 2$, *then* $|C : C \cap X| \leq 2|\mathbb{F}|^k$ *resp.* $|(C \cap H) : (C \cap H \cap X)| \leq 2|\mathbb{F}|^k$.

Proof: Suppose first that p is odd. It suffices to show that Z is contained in X. To this end we will show that $Z \leq R'$, because $R' \leq ((X \cap C)Z)' \leq X'$.

There exists a non-degenerate subspace W_1 in V such that $W = W_0 \oplus W_1$. By Lemma 4.2, W_0 is contained in a non-degenerate subspace $S \leq V$, which is an orthogonal sum of $\dim_{\mathbb{F}} W_0$ hyperbolic planes. And so $S = W_0 \oplus S_0$ for some totally isotropic subspace S_0 of S. Because $W_1 \leq W_0^{\perp}$, the space S has trivial intersection with W_1. Let $U = W_0 \oplus H \oplus S_0$, where H is a hyperbolic plane in W_1. As in the second paragraph of proof of part (b) of Theorem A, $W = (W \cap U) \oplus (W \cap U^{\perp})$, and every element in the stabilizer R_0 of the series $U > W \cap U > W_0 > 0$ in U extends to an element in R which centralizes $U^{\perp}$.

From Lemma 5.5, the group Z centralizes $U^{\perp}$. It therefore suffices to show that the restriction to U of any element from Z is a commutator in R_0. In order to better understand the structure of R_0, we will write its elements now as block matrices and make some explicit calculations. For convenience we arrange the blocks in an order corresponding to W_0, H, S_0. Then elements of R_0 take the form

$$\begin{pmatrix} I & A & B \\ 0 & I & D \\ 0 & 0 & I \end{pmatrix}.$$

With respect to a suitable basis, the Gram matrix of b has the form

$$\begin{pmatrix} 0 & 0 & I \\ 0 & J & 0 \\ \pm I & 0 & 0 \end{pmatrix}, \quad \text{where } J \text{ is the } (2 \times 2)\text{-matrix } \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

The elements of R_0 are characterized by the condition

$$\begin{pmatrix} I & 0 & 0 \\ A^t & I & 0 \\ B^t & D^t & I \end{pmatrix} \begin{pmatrix} 0 & 0 & I \\ 0 & J & 0 \\ \pm I & 0 & 0 \end{pmatrix} \begin{pmatrix} I & A & B \\ 0 & I & D \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 & I \\ 0 & J & 0 \\ \pm I & 0 & 0 \end{pmatrix}.$$

This is equivalent to $D^t J \pm A = 0$ and $B^t \pm B + D^t J D = 0$. And so any commutator

$$\left[\begin{pmatrix} I & A & B \\ 0 & I & D \\ 0 & 0 & I \end{pmatrix}, \begin{pmatrix} I & X & Y \\ 0 & I & Z \\ 0 & 0 & I \end{pmatrix} \right] = \begin{pmatrix} I & 0 & AZ - XD \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

of two elements in R satisfies $AZ - XD = D^t J Z \pm Z^t J D$.

We want to show now that, over the field $\mathbb{F}$ of odd characteristic, the commutator subgroup R_0' contains the central subgroup $C_{R_0}(U/W_0) \cap C_{R_0}(W \cap U)$ of all matrices

$$\begin{pmatrix} I & 0 & T \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

where T is (anti)symmetric. To this end, consider the elementary matrices $D = E_{1i}$ and $Z = E_{2j}$. Choose $A = D^tJ$ and $X = Z^tJ$, and let $B = 0 = Y$. These choices yield matrices in R_0, since $J = E_{12} \pm E_{21}$ gives

$$B^t \pm B + D^tJD = D^tJD = E_{i1}E_{12}E_{1i} \pm E_{i1}E_{21}E_{1i} = 0,$$

and similarly $Y^t \pm Y + Z^tJZ = 0$. Now

$$D^tJZ = E_{i1}E_{12}E_{2j} \pm E_{i1}E_{21}E_{2j} = E_{i2}E_{2j} = E_{ij},$$

whence

$$D^tJZ \pm Z^tJD = D^tJZ \mp (D^tJZ)^t = E_{ij} \mp E_{ji}.$$

In the case when $p = 2$, the same argument shows that R_0' contains every element from Z, whose matrix entry T has zero diagonal. ∎

Note also that $C = C_G(W^\perp)$ and $N_G(W) = N_G(W^\perp)$ whenever $W = W^{\perp\perp}$ (Lemma 7.1).

Corollary 7.7: *If $V = W \oplus W^\perp$, then $C_G(W^\perp) \leq X \leq N_G(W^\perp)$ resp. $C_H(W^\perp) \leq X \cap H \leq N_H(W^\perp)$.*

Proof: From Lemmata 7.2 and 7.3, $C_G(W^\perp)$ is the classical finitary linear group with natural module W. ∎

Proof of Theorem B: From Corollary 5.2, there exist a minimal X-invariant subspace W of finite codimension in V, and a maximal finite-dimensional X-invariant subspace $W_0 \leq W$. Let

$$N = N_G(W) \cap N_G(W_0) \quad \text{and} \quad C = C_G(V/W) \cap C_G(W_0).$$

Let $R = C_C(W/W_0)$ denote the unipotent radical of N. Let $W_1 \leq V$ be a subspace of dimension $\dim_{\mathbb{F}} W_0 + \dim_{\mathbb{F}} V/W$ which contains W_0 and supplements W in V. Then there exists a finite-dimensional $\Delta_1 \leq \Delta$ such that $V = W_1 \oplus W_2$ where $W_2 = \Delta_1^\perp$. Now V and Δ can be written as unions of local systems of finite-dimensional subspaces V_i resp. Γ_i $(i \in I)$ such that $V = V_i \oplus \Gamma_i^\perp$ and

$W_1 \leq V_i$, $\Delta_1 \leq \Gamma_i$ for all i. Let $G_i = N_G(V_i) \cap C_G(\Gamma_i^\perp) = \mathrm{T}_{\mathbb{F}}(\Gamma_i, V_i)$. Clearly, $C_{G_i}(W_1) \cap N_{G_i}(V_i \cap W_2) = \mathrm{T}_{\mathbb{F}}(\Gamma_i \cap W_1^\perp, V_i \cap W_2)$ is a complement to $R \cap G_i$ in $C \cap G_i$. Therefore $Y = \mathrm{T}_{\mathbb{F}}(W_1^\perp, W_2)$ is a complement to R in C. In particular, C acts like a special transvection group on $\widetilde{W} = W/W_0$. Now Lemma 5.4 and Propositions 5.6/5.12 imply that X too acts like a special transvection group on $\widetilde{W}$, and that $\widetilde{W}$ is the natural or the conatural module for this action.

For every $x \in R$, homomorphisms $\tau_x\colon \widetilde{W} \longrightarrow W_0$ and $\sigma_x\colon V/W \longrightarrow \widetilde{W}$ are given via

$$\widetilde{w}\tau_x = [w, x] \quad \text{for all} \quad w \in W \quad \text{and}$$
$$(v + W)\sigma_x = [v, x] + W_0 \quad \text{for all} \quad v \in V.$$

Clearly N acts on R via conjugation, and actions of N on $\mathrm{Hom}(\widetilde{W}, W_0)$ resp. on $\mathrm{Hom}(V/W, \widetilde{W})$ are given by $\varphi^g = g^{-1}\varphi g$ for all $g \in N$, and all φ in $\mathrm{Hom}(\widetilde{W}, W_0)$ resp. in $\mathrm{Hom}(V/W, \widetilde{W})$. Now the map $\mu\colon x \mapsto (\tau_x, \sigma_x)$ is a homomorphism of N-groups, with kernel $Z = C_R(W) \cap C_R(V/W_0)$. In particular, R/Z is isomorphic to an N-submodule of $\mathrm{Hom}(\widetilde{W}, W_0) \oplus \mathrm{Hom}(V/W, \widetilde{W})$, whence $R' \leq Z$.

Now $\mathrm{Hom}(\widetilde{W}, W_0)$ is isomorphic to a direct sum of finitely many copies of the dual module $\widetilde{W}^*$, and $[\widetilde{W}^*, X^o]$ is an irreducible X^oR/R-module, hence natural or conatural by [22], Theorem B. Moreover, $\mathrm{Hom}(V/W, \widetilde{W})$ is isomorphic to a direct sum of finitely many copies of the (co)natural X^oR/R-module $\widetilde{W}$. Therefore Lemma 6.2 and [21], Theorem 2 yield that $\overline{X}^o$ splits over $\overline{R}$, with some complement $\overline{Y}$. Moreover, Z is isomorphic to a submodule of $\mathrm{Hom}(V/W, W_0)$, and hence finite. It now follows as in the proof of part (b) of Theorem A that $Y = Y' \times Z$ and that Y' is a complement to R in X^o.

From Proposition 6.3 we obtain $W = [W, Y'] \oplus W_0$ as Y'-modules, and hence $U = [W, Y'] = [V, Y']$ is an irreducible Y'-submodule of finite codimension in V. By [22], Theorem B, this is either the natural or the conatural Y'-module. Correspondingly, $\Gamma = [\Delta, Y']$ is an irreducible Y'-submodule of finite codimension in Δ, and this must then be the conatural resp. the natural Y'-module. Altogether $Y' = \mathrm{T}_{\mathbb{F}}(\Gamma, U)$ resp. $Y' = \mathrm{T}_{\mathbb{F}}(U, \Gamma)$. Here $\mathrm{ann}_\Gamma U = 0$ and $\mathrm{ann}_U \Gamma = 0$, because Γ and U are irreducible Y'-modules. ∎

8. Some consequences

In this section we essentially discuss some of the problems raised by A. E. Zalesskiĭ in [30], pp. 223–224.

Clearly no confined subgroup in a classical finitary linear group G is locally soluble, because the centralizer of every finite-dimensional subspace of the natural G-module contains an infinite simple section.

PROPOSITION 8.1: *Let G be a classical finitary linear group of isometries, with natural module V over the finite field $\mathbb{F}$. Then*

(a) *every confined subgroup of G has just finitely many overgroups in G, and*
(b) *the intersection of two confined subgroups of G is again confined in G.*

Proof: (a) Suppose first that G is a classical finitary linear group of isometries, which is not symplectic in case $\operatorname{char}\mathbb{F} = 2$. Assume that X is a confined subgroup in G with infinitely many overgroups X_i $(i \in I)$. From Corollary 5.2 there exists a minimal X-invariant subspace U of finite codimension in V. Also, there exists a minimal X_i-invariant subspace U_i of finite codimension in V for each i, and X_i has finite index in $N_G(U_i)$ from Theorem A. By minimality, U must be contained in every U_i. But the field $\mathbb{F}$ is finite, and so we may assume without loss that all the U_i coincide with a single subspace W of V. On the other hand, it follows from Proposition 7.6 that $N_G(W)$ has just finitely many subgroups of finite index, a contradiction. Hence X admits just finitely many overgroups in G.

Suppose now that G is a finitary symplectic group and $\operatorname{char}\mathbb{F} = 2$, and make the above assumption. From Theorem A there exist quadratic forms q resp. q_i on V with associated orthogonal form b, and minimal X- resp. X_i-invariant subspaces U resp. U_i of finite codimension in V, such that $X \cap H$ resp. $X_i \cap H_i$ has finite index in $N_H(U)$ resp. $N_{H_i}(U_i)$; here H resp. H_i denotes the finitary orthogonal group of isometries of q resp. q_i in G. Again we may assume that all the U_i coincide with a single subspace W of U. The radical of W is complemented by a non-degenerate subspace $\overline{W}$ of finite codimension in W. By Lemma 7.3, the derived subgroup of $C_{X\cap H}(V/\overline{W})$ resp. $C_{X\cap H_i}(V/\overline{W})$ induces on $\overline{W}$ the finitary orthogonal group with respect to the form q resp. q_i. Now Lemmata 4.2 and 5.10 imply that the forms q_i coincide with q on $\overline{W}$. Since $\overline{W}$ has finite codimension in V, and since the field $\mathbb{F}$ is finite, there exist just finitely many possibilities to extend $q|_{\overline{W}}$ to a quadratic form on V with associated bilinear form b. Hence we may assume without loss, that the forms q_i, and also the subgroups H_i, coincide.

It therefore suffices to show that $N_G(W)$ has just finitely many subgroups X_i such that W is a minimal X_i-invariant subspace of finite codimension in V, and such that $X_i \cap H_i$ has finite index in $N_{H_i}(W)$. We stick to the notation introduced in Section 7. By Lemma 7.4, every vector in W_0 is singular with respect to q_i. Therefore the unipotent radical R of $C = C_G(V/W)$ is contained in H_i. Recall that C induces $\mathrm{FSp}_{\mathbb{F}}(\widetilde{W}, \widetilde{b})$ on $\widetilde{W}$, while $X_i Z$ induces either $\mathrm{FSp}_{\mathbb{F}}(\widetilde{W}, \widetilde{b})$, or $(C \cap H_i)/R$, or $(C \cap H_i)'R/R$ on $\widetilde{W}$. In the first case, $C' \leq X_i$, and X_i is one of the finitely many subgroups of finite index in $N_G(W)$. In the second case, X_iZ/R contains the direct product of $(C \cap H_i)'R/R$ with a subgroup of the finite group $\big(N_G(V/W)/C_G(V/W)\big) \times \big(N_G(W_0)/C_G(W_0)\big)$. Also in this situation there are just finitely many possibilities for X_iZ, hence also for the overgroup X_i of the finite residual of X_iZ.

(b) Let X_1 and X_2 be confined subgroups in G. If G is not symplectic in case $\operatorname{char}\mathbb{F} = 2$, then there exist subspaces W_1 and W_2 in V such that $C_G(V/W_i) \cap X_i$ has finite index in $C_G(V/W_i)$ for $i = 1, 2$. Then $X_1 \cap X_2 \cap C_G(V/(W_1 \cap W_2))$ has finite index in $C_G(V/(W_1 \cap W_2)) = C_G(V/W_1) \cap C_G(V/W_2)$, whence $X_1 \cap X_2$ is confined in G by Proposition 4.6.

Suppose now that G is a finitary symplectic group and $\operatorname{char}\mathbb{F} = 2$. Then there are quadratic forms q_1 and q_2 on the natural $\mathbb{F}G$-module, whose associated bilinear form is the symplectic form on V, such that $X_i \cap C_{H_i}(V/W_i)$ has finite index in $C_{H_i}(V/W_i)$, where H_i is the finitary orthogonal group of isometries of q_i. As in part (a), there exists a non-degenerate subspace $\overline{W}$ of finite codimension in $W_1 \cap W_2$. Let U be the largest subspace of $\overline{W}$ on which q_1 and q_2 coincide. For every $\lambda \in \mathbb{F}$, the set of all $w \in \overline{W}$ with $q_1(w) - q_2(w) = \lambda$ is a coset of U in $\overline{W}$. Therefore U has codimension 1 in $\overline{W}$. But now $Y = X_1 \cap X_2 \cap C_{H_i}(V/U)$ has finite index in $N_{H_i}(U)$, whence Y and $X_1 \cap X_2$ are confined in G by Proposition 4.7. ∎

A subgroup X of the locally finite group G is said to be **cobounded** in G, if there exists a local system of finite subgroups G_i $(i \in I)$ in G such that the numbers

$$\beta_i(t, g) = \frac{|G_i \cap t^G|}{1 + |G_i \cap X^g \cap t^G|}$$

are uniformly bounded for all $i \in I$ and all $t, g \in G$; here t^G denotes the conjugacy class of t in G. Cobounded subgroups are supposed to be large, and in fact, every cobounded subgroup of a locally finite, simple group G is confined in G ([26], Proposition 2). It was conjectured in [26] and in [30], that the converse holds too. This converse holds in the case of alternating groups ([26], Theorem 3). Although we cannot decide the conjecture for classical finitary linear groups, we shall now give evidence that the above notion of coboundedness seems to be a bit too strong to be satisfied by confined subgroups.

PROPOSITION 8.2: *Let G be a countable finitary symplectic group with natural module V over the finite field with q elements. Consider the confined subgroup $X = C_G(W)$ of G, where W is a non-degenerate $2k$-dimensional subspace of*

V. Let V be the union of the ascending chain V_{2n} ($n \in \mathbb{N}$) of non-degenerate subspaces of dimensions $2n$, and let $G_{2n} = C_G(V_{2n}^{\perp})$ form the 'natural' local system in G. Then for every $r \in \mathbb{N}$, there exists an element $t_r \in G$ such that $\beta_{2n}(t_r, g)$ is asymptotically equal to $q^{2k(r+1)}$ for every $g \in G$, as n tends to infinity.

Proof: Let U_i ($i \in \mathbb{N}$) be pairwise orthogonal hyperbolic planes in V. Let τ_i be transvections in G with $[V, \tau_i] \leq U_i$ and $U_i^{\perp} \leq C_V(\tau_i)$. We choose $t_r = \tau_1 \dots \tau_{r+1}$.

Let us determine $|G_{2n} \cap \tau_1^G|$. To this end we may assume without loss that $U_1 \leq V_{2n}$. For any transvection $\tau \in G_{2n}$, an ordered hyperbolic pair in V_{2n} is given by x, v where $0 \neq x \in [V, \tau]$ and $v \in V_{2n} \smallsetminus [V, \tau]^{\perp}$ with $b(x, v) = 1$. Here, two ordered hyperbolic pairs x, v and y, w are obtained from the same transvection if and only if $x = \lambda y$ and $v + [V, \tau]^{\perp} = \lambda^{-1} w + [V, \tau]^{\perp}$ for some scalar $\lambda \neq 0$. The number of ordered hyperbolic pairs in V_{2n} is $(q^{2n} - 1)q^{2n-1}$. Therefore the number of transvections in G_{2n} is $(q^{2n} - 1)/(q - 1)$. However there are $q - 1$ different conjugacy classes of transvections in G_{2n} of equal size. Altogether,

$$|G_{2n} \cap \tau_1^G| = |G_{2n} \cap \tau_1^{G_{2n}}| = \frac{q^{2n} - 1}{(q-1)^2} = f(n).$$

Since $\tau_2 \dots \tau_{r+1} \in C_G(U_1) \cap N_G(U_1^{\perp})$, an easy recursion now shows that

$$|G_{2n} \cap t_r^G| = \frac{1}{(r+1)!} \cdot f(n) \cdot \ldots\ldots \cdot f(n-r) \quad \text{for all sufficiently large } n.$$

The subspace W is non-degenerate, and so X^g is the finitary symplectic group on $(Wg)^{\perp}$. It follows that

$$|G_{2n} \cap X^g \cap t_r^G| = \frac{1}{(r+1)!} \cdot f(n-k) \cdot \ldots\ldots \cdot f(n-k-r)$$

for all sufficiently large n. Thus,

$$\begin{aligned} \frac{|G_{2n} \cap t_r^G|}{1 + |G_{2n} \cap X^g \cap t_r^G|} &= \frac{f(n) \dots f(n-r)}{(r+1)! + f(n-k) \dots f(n-k-r)} \\ &= \frac{(q^{2n} - 1) \dots (q^{2(n-r)} - 1)}{(r+1)! \cdot (q-1)^{2(r+1)} + (q^{2(n-k)} - 1) \dots (q^{2(n-k-r)} - 1)} \end{aligned}$$

for large n. Here the highest exponent of q takes the value

$$\begin{aligned} 2\left(\sum_{\nu=n-r}^{n} \nu - \sum_{\nu=n-k-r}^{n-k} \nu \right) &= 2 \sum_{\nu=0}^{r} \big((n - r + \nu) - (n - r - k + \nu) \big) \\ &= 2k(r+1). \qquad \blacksquare \end{aligned}$$

Note that Proposition 8.2 does not show that centralizers X of finite-dimensional non-degenerate subspaces in countable finitary symplectic groups are not cobounded, since the local system is chosen in a very specific way. It might be possible that a slick choice of local system turns X into a cobounded subgroup. However, the authors believe that X is not cobounded here.

Proposition 8.2 rather suggests to introduce the following notion. A subgroup X of the locally finite group G is said to be **weakly cobounded** in G, if there exists a local system of finite subgroups G_i ($i \in I$) in G such that for every fixed $t \in G$, the numbers $\beta_i(t, g)$ are bounded for all $i \in I$ and all $g \in G$. Note that the proof of [26], Proposition 2 shows as well that every weakly cobounded subgroup of a locally finite, simple group is confined.

CONJECTURE 8.3: *Every confined subgroup of a locally finite, simple group is weakly cobounded.*

We close with an interesting observation concerning joins of centralizers in classical finitary linear groups of isometries.

PROPOSITION 8.4: *Let G be a classical finitary linear group of isometries, with natural module V over the finite field $\mathbb{F}$. If U_1 and U_2 are finite-dimensional subspaces of V, then $J = \langle C_G(U_1), C_G(U_2)\rangle$ has finite index in $C_G(U_1 \cap U_2)$. In particular, $|C_G(U_1 \cap U_2) : J| \leq 2$ whenever $\operatorname{char}\mathbb{F}$ is odd. And both centralizers are equal when $U_1 \cap U_2$ is non-degenerate, or when G is symplectic or unitary in odd characteristic.*

Proof: From Lemma 4.4, Proposition 4.6, and from Lemma 7.1, the subgroup J is confined in G. Hence there exists a minimal J-invariant subspace W of finite codimension in V (Corollary 5.2). Since $U_i^\perp$ is the smallest $C_G(U_i)$-invariant subspace of finite codimension in V, the space $U = U_1^\perp + U_2^\perp$ must be contained in W. On the other hand, Lemma 7.1 gives $C_G(U_i) = C_G(V/U_i^\perp)$, whence U is J-invariant. It follows that $U = W$.

Clearly, $J \leq C_G(U_1 \cap U_2) \leq N_G((U_1 \cap U_2)^\perp) = N_G(U)$ by Lemma 4.4. If G is not a finitary symplectic group, or if $\operatorname{char}\mathbb{F}$ is odd, then J has finite index in $N_G(U)$ and in $C_G(U_1 \cap U_2)$ by virtue of Theorem A. If G is finitary symplectic in characteristic 2, then we consider a non-degenerate subspace V_i of finite codimension in $U_i^\perp$. Now the subgroup $C_G(V/V_i)$ of $C_G(U_i)$ does not respect any non-zero quadratic form related to the given symplectic form on V_i. Therefore J must induce the finitary symplectic group on the quotient $\widetilde{W}$, and J has finite index in $N_G(W)$ too by Theorem A.

The remaining assertions follow from Proposition 7.6 and Corollary 7.7. ∎

Presumably $\langle C_G(U_1), C_G(U_2)\rangle = C_G(U_1 \cap U_2)$ always holds in the situation of Proposition 8.4. But we do not pursue this further here since the proof would require an even more detailed study of the structure of $C_G(U_1 \cap U_2)$.

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